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TESTING EFFICIENCY HYPOTHESES IN JOINT PRODUCTION: A PARAMETRIC APPROACH

C. A. Knox Lovell and Robin C. Sickles*

I. Introduction

IN recent years a great deal of research has been directed to the modelling and measurement of technical and allocative efficiency in production. With few exceptions this research has been restricted to single-product firms.[1] However, recent developments in duality theory have facilitated the extension of this research to multi-product firms. The main purpose of this paper is to develop a model of the multiproduct firm in which the possibilities of both technical and allocative inefficiency are incorporated in an econometrically useful way. The first model we develop includes a nonneutral[2] type of technical inefficiency and three distinguishable types of allocative inefficiency—output mix, input mix, and scale. Each type of inefficiency is costly to the firm, in the sense that each causes a reduction in profit beneath the maximum value attainable under full efficiency. The cost of each type of inefficiency depends on the magnitude of the inefficiency and the structure of the underlying production technology. In the second model we develop, technical inefficiency remains nonneutral, but allocative inefficiency is not generally decomposable into output mix, input mix and scale components. However, both technical and allocative inefficiency remain costly to the firm, the cost of each type of inefficiency depending on its magnitude and the structure of the underlying production technology.

We model the technology of a competitive profit maximizing multi-product firm with the dual profit function. This enables us to use Hotelling's Lemma to generate a system of profit maximizing output supply and input demand equations. These equations are then modified to allow for the possibility of technical and three types of allocative inefficiency. A virtue of using the profit function to represent production technology is that it permits a straightforward comparison of maximum profit under full efficiency with actual profit, and with the profit that would result from any combination of the four types of inefficiency. This enables us to allocate the cost of inefficiency to each of four components. Our model of inefficiency is parametric, and is embedded in a Generalized Leontief profit function, although any flexible specification of the profit function can be used.

The model is developed in sections II–IV. Estimation of the model is considered in section V. An empirical example designed to illustrate the workings of the model is discussed in section VI. Section VII concludes.

II. Efficient Production Technology

We consider a production unit employing inputs $x = (x_1, \ldots, x_n) \geqq 0$ to produce outputs $y = (y_1, \ldots, y_m) \geqq 0$. The set of all technologically feasible input-output vectors is given by the *production possibilities set T*, which is assumed to satisfy the following regularity conditions:

- T.1: T is a nonempty subset of Ω^{m+n}, and if $(y, -x) \in T$ then $y \geqq 0$, $x \geqq 0$;
- T.2: T is a closed set which is bounded from above;
- T.3: T is a convex set;
- T.4: If $(y, -x) \in T$, then $(y', -x') \in T$ for all $0 \leqq y' \leqq y$, $x' \geqq x$.

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[1] The extant research is surveyed by Førsund, Lovell and Schmidt (1980). Research on multi-product firms has touched on inefficiency in a peripheral way, typically by investigating the appropriateness of the firm's output mix. An example of this research is the investigation of the structure of cost in U.S. railroads by Brown, Caves and Christensen (1979). The operations research literature contains some recent studies of efficiency in multi-product firms, although these studies investigate technical efficiency only. For examples see Charnes, Cooper and Rhodes (1978) and Färe and Grosskopf (1981).

[2] The most frequently used term is "radial," but that is a misnomer in the multiple output, multiple input case. Scaling up all outputs and scaling down all inputs by the same proportion (i.e., neutrally) traces out a hyperbola rather than a ray.

We now assume that the production unit takes output prices $p = (p_1, \ldots, p_m) > 0$ and input prices $w = (w_1, \ldots, w_n) > 0$ as given, and attempts to adjust outputs and inputs so as to solve

$$\sup_{y,x} \{py - wx: (y, -x) \in T\}.$$

If $(y^0, -x^0)$ solves this problem then the production unit's *profit function* is $\pi(p,w) \equiv py^0 - wx^0$, where π satisfies the following regularity conditions:

- π.1: $\pi(p,w)$ is a real valued function defined for all $(p,w) > 0$;
- π.2: $\pi(p,w)$ is nondecreasing in p and nonincreasing in w;
- π.3: $\pi(\lambda p, \lambda w) = \lambda\pi(p,w)$ for all $\lambda > 0$;
- π.4: $\pi(p,w)$ is a convex function in (p,w).

The usefulness of the profit function results from two facts. First, there exists a duality relationship between a production possibilities set T satisfying {T.1–T.4} and a profit function π satisfying {π.1–π.4}, and so π and T provide equivalent representations of the technology of a profit maximizing production unit.[3] Second, *Hotelling's Lemma* states that profit maximizing output supply and input demand equations can be obtained directly from the profit function by means of

$$\nabla_p \pi(p,w) = y(p,w), \quad \nabla_w \pi(p,w) = -x(p,w),$$

at all $(p,w) > 0$ for which $\pi(p,w)$ is differentiable. Properties of these supply and demand equations are inherited directly from properties {π.1–π.4} of the profit function.

To illustrate, suppose that π is of the Generalized Leontief form (Diewert, 1971), with $m = n = 2$ for concreteness. Then,

$$\begin{aligned}\pi(p,w) = {} & A_{11}p_1 + A_{12}p_1^{1/2}p_2^{1/2} + A_{13}p_1^{1/2}w_1^{1/2} \\ & + A_{14}p_1^{1/2}w_2^{1/2} + A_{21}p_2^{1/2}p_1^{1/2} \\ & + A_{22}p_2 + A_{23}p_2^{1/2}w_1^{1/2} \\ & + A_{24}p_2^{1/2}w_2^{1/2} + A_{31}w_1^{1/2}p_1^{1/2} \\ & + A_{32}w_1^{1/2}p_2^{1/2} + A_{33}w_1 \\ & + A_{34}w_1^{1/2}w_2^{1/2} + A_{41}w_2^{1/2}p_1^{1/2} \\ & + A_{42}w_2^{1/2}p_2^{1/2} + A_{43}w_2^{1/2}w_1^{1/2} \\ & + A_{44}w_2\end{aligned}$$

where $A_{ij} = A_{ji}$ for all $j \neq i$. Hotelling's Lemma yields the profit maximizing output supply and input demand equations:

$$\begin{aligned}y_1(p,w) &= A_{11} + A_{12}(p_1/p_2)^{-1/2} \\ &\quad + A_{13}(p_1/w_1)^{-1/2} + A_{14}(p_1/w_2)^{-1/2}, \\ y_2(p,w) &= A_{22} + A_{21}(p_1/p_2)^{1/2} + A_{23}(p_2/w_1)^{-1/2} \\ &\quad + A_{24}(p_2/w_2)^{-1/2}, \\ -x_1(p,w) &= A_{33} + A_{31}(p_1/w_1)^{1/2} \\ &\quad + A_{32}(p_2/w_1)^{1/2} + A_{34}(w_1/w_2)^{-1/2}, \\ -x_2(p,w) &= A_{44} + A_{41}(p_1/w_2)^{1/2} \\ &\quad + A_{42}(p_2/w_2)^{1/2} + A_{43}(w_1/w_2)^{1/2}.\end{aligned}$$

Although the Generalized Leontief specification of π satisfies π.1 and π.3 by construction, it leaves monotonicity (π.2) and convexity (π.4) as hypotheses to be tested.

III. Inefficiency

We now incorporate inefficiency into the model. The production unit is said to be *technically inefficient* if it operates on the interior of its production possibilities set, so that for observed input-output vector $(y, -x) \in T$ there exists $(y', -x') \in T$ such that $(y', -x') \geq (y, -x)$. Since $(py' - wx') > (py - wx)$, technical inefficiency leads to a failure to maximize profit. The production unit is said to be *allocatively inefficient* if it operates at the wrong point on the boundary of its production possibilities set, given the output and input prices it faces and given its behavioral objective of profit maximization. Allocative inefficiency also leads to a failure to maximize profit.

The Generalized Leontief system of output supply and input demand equations can be modified to incorporate inefficiency in the following way. Technical inefficiency is modelled by adjusting the intercepts so as to permit a divergence between actual and profit maximizing output supplies and input demands. Allocative inefficiency is modelled, following Toda (1976) and Atkinson and Halvorsen (1980), by assuming that the production unit adjusts output supplies and input demands to the wrong price ratios. Thus,

[3] The profit function is discussed in Diewert (1973) and McFadden (1978). As Diewert has pointed out, if T satisfies only T.1 and T.2, the derived function π still satisfies {π.1–π.4}. In this case π is dual to the convex free disposal hull T^* of T. Thus if technology is characterized by regions of increasing returns to scale, or only weak disposability, these properties will not show up in the derived profit function. Only when all parts of {T.1–T.4} hold does π completely characterize T.

$$y_1(p,w,\phi,\theta) = (A_{11} - \phi_1) + A_{12}[\theta_{12}(p_1/p_2)]^{-1/2} + A_{13}[\theta_{13}(p_1/w_1)]^{-1/2} + A_{14}[\theta_{14}(p_1/w_2)]^{-1/2},$$

$$y_2(p,w,\phi,\theta) = (A_{22} - \phi_2) + A_{12}[\theta_{12}(p_1/p_2)]^{1/2} + A_{23}[\theta_{23}(p_2/w_1)]^{-1/2} + A_{24}[\theta_{24}(p_2/w_2)]^{-1/2},$$

$$-x_1(p,w,\phi,\theta) = (A_{33} - \phi_3) + A_{13}[\theta_{13}(p_1/w_1)]^{1/2} + A_{23}[\theta_{23}(p_2/w_1)]^{1/2} + A_{34}[\theta_{34}(w_1/w_2)]^{-1/2},$$

$$-x_2(p,w,\phi,\theta) = (A_{44} - \phi_4) + A_{14}[\theta_{14}(p_1/w_2)]^{1/2} + A_{24}[\theta_{24}(p_2/w_2)]^{1/2} + A_{34}[\theta_{34}(w_1/w_2)]^{1/2}.$$

In this modified Generalized Leontief system the parameters $\phi_i \geqq 0$ measure the underproduction of outputs $(i = 1,2)$ and the excessive usage of inputs $(i = 3,4)$ attributable to technical inefficiency. The components of ϕ are expressed in the same units as are the components of $(y, -x)$, and can be converted into percentages if necessary. Thus the parameter vector ϕ provides a nonneutral measure of technical inefficiency, even if all components of ϕ are equal. This is in contrast to the Debreu (1951)–Farrell (1957) measure, which is neutral.[4] The parameters $\theta_{ij} > 0, j > i$, are pure numbers that measure the ratio of perceived to actual price ratios. They capture what Schmidt and Lovell (1979) refer to as the systematic component of allocative inefficiency. Thus $\theta_{ij} \gtreqqless 1$ implies that the perceived price ratio exceeds, equals, or falls below the actual price ratio, so that the corresponding commodity mix is inefficiently small, allocatively efficient, or inefficiently large.

The cost of each type of inefficiency depends on output and input prices and the structure of technology, in addition to the magnitudes of the parameters ϕ and θ. In the presence of both technical and allocative inefficiency the observed value of profit can be expressed

$$\pi(q, \phi, \theta) = \sum_{i=1}^{4} (A_{ii} - \phi_i)q_i + \sum_{i=1}^{3} \sum_{\substack{j=2 \\ j>i}}^{4} A_{ij}(\theta_{ij}^{-1/2} + \theta_{ij}^{1/2})q_i^{1/2}q_j^{1/2}$$

where $q \equiv (p_1, p_2, w_1, w_2)$. The effect of technical inefficiency alone on profit is given by the difference

$$\pi(q) - \pi(q, \phi) = \sum_{i=1}^{4} \phi_i q_i,$$

which is zero if $\phi_i = 0$, for all i, and positive otherwise. The effect of allocative inefficiency alone on profit is given by the difference

$$\pi(q) - \pi(q, \theta) = \sum_{i=1}^{3} \sum_{\substack{j=2 \\ j>i}}^{4} A_{ij}q_i^{1/2}q_j^{1/2} \times [2 - (\theta_{ij}^{-1/2} + \theta_{ij}^{1/2})],$$

which is zero if all $\theta_{ij} = 1$. If any $\theta_{ij} \neq 1$, then $[\pi(q) - \pi(q, \theta)] \geqq 0$ by virtue of the convexity property $\pi.4$, with equality holding if and only if the corresponding $A_{ij} = 0$.

Allocative inefficiency and its effect on profit can be decomposed. If $\theta_{12} \neq 1$, all other $\theta_{ij} = 1$, outputs are allocated according to the wrong output price ratio, and

$$\pi(q) - \pi(q, \theta_{12}) = A_{12}q_1^{1/2}q_2^{1/2}[2 - (\theta_{12}^{-1/2} + \theta_{12}^{1/2})].$$

Thus *output mix inefficiency* $(\theta_{12} \neq 1)$ leads to a failure to maximize profit if $A_{12} < 0$, and has no effect on profit if $A_{12} = 0$. Similarly, if $\theta_{34} \neq 1$, all other $\theta_{ij} = 1$, inputs are allocated according to the wrong input price ratio, and

$$\pi(q) - \pi(q, \theta_{34}) = A_{34}q_3^{1/2}q_4^{1/2} \times [2 - (\theta_{34}^{-1/2} + \theta_{34}^{1/2})].$$

Thus *input mix inefficiency* $(\theta_{34} \neq 1)$ leads to a failure to maximize profit if $A_{34} < 0$, and has no effect on profit if $A_{34} = 0$. Finally, if $(\theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}) \neq (1, 1, 1, 1)$ with $\theta_{12} = \theta_{34} = 1$, inputs and outputs are allocated according to the wrong input-output price ratios, with too few (too many) inputs being used to produce too little (too much) output. In this case

$$\pi(q) - \pi(q, \theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}) = \sum_{i=1}^{2} \sum_{j=3}^{4} A_{ij}q_i^{1/2}q_j^{1/2}[2 - (\theta_{ij}^{-1/2} + \theta_{ij}^{1/2})].$$

Thus *scale inefficiency* leads to a failure to maximize profit if all components of $(A_{13}, A_{14}, A_{23}, A_{24})$ corresponding to non-unitary components of $(\theta_{13}, \theta_{14}, \theta_{23}, \theta_{24})$ are nonzero. Scale

[4] In a log-linear system such as translog the components of ϕ measure technical inefficiency in percentage terms, and neutrality holds if, and only if, all components of ϕ are equal.

inefficiency has no effect on profit if the components of $(A_{13}, A_{14}, A_{23}, A_{24})$ corresponding to non-unitary components of $(\theta_{13}, \theta_{14}, \theta_{23}, \theta_{24})$ are zero.

This approach to the measurement of the magnitudes and costs of technical and allocative inefficiency is illustrated for a single output, single input production unit in figure 1. A technically and allocatively inefficient production unit operates at point A on the interior of its production possibilities set T. It would be technically (but not allocatively) efficient if it used less (ϕ_3) of input X_1 to produce more (ϕ_1) of output y_1 at point B on the boundary of T. It would also earn greater profit, although not maximum profit, since point B remains allocatively inefficient. It would be both technically and allocatively efficient, and earn maximum profit, if it altered its scale of operation by moving to point C on the boundary of T. Since $0 < \theta < 1$, the movement from B to C involves an increase in the scale of operation. Note finally that in a single output, single input case all allocative inefficiency is scale inefficiency.

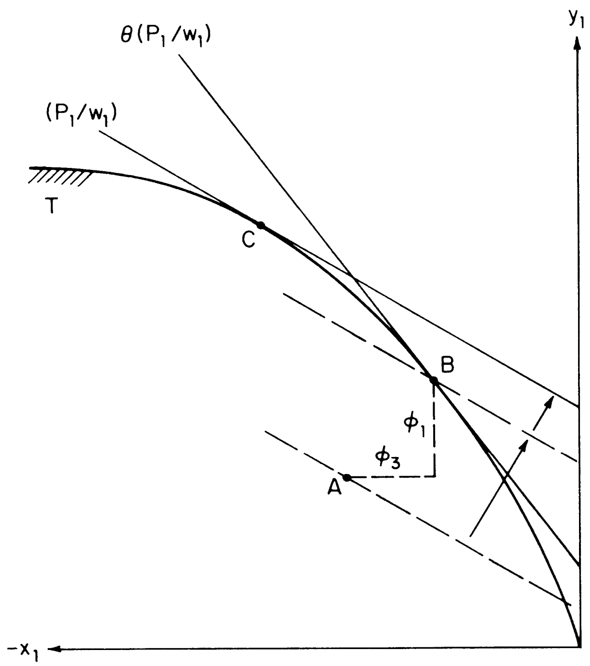
IV. Consistent Allocative Inefficiency

Thus far we have ignored the fact that the production unit faces only four market prices, and only three independent market price ratios, although we have used six independent θ_{ij}'s to model allocative inefficiency. Clearly the *market* price ratios can be expected to be consistent, in the sense that any three independent price ratios can be used to determine the remaining three price ratios. What is not so clear is whether the *perceived* price ratios as modelled by $[\theta_{ij}(q_i/q_j)]$ can be expected to be consistent also. That is, it is unclear whether or not an allocatively inefficient production unit can be expected to be consistent in its misperception of market price ratios. The preceding analysis, with three independent market price ratios and six independent perceived price ratios, permits inconsistent allocative inefficiency.

Consistent allocative inefficiency can be modelled as a constrained version of the preceding model. We simply constrain perceived price ratios to satisfy

$$[\theta_{ij}(q_i/q_j)] \cdot [\theta_{jk}(q_j/q_k)] = \theta_{ik}(q_i/q_k), \quad i < j < k,$$

FIGURE 1.—TECHNICAL AND ALLOCATIVE INEFFICIENCY ILLUSTRATED



which given consistency of market price ratios requires that

$$\theta_{ik} = \theta_{ij} \cdot \theta_{jk}, \quad i < j < k,$$

which reduces the number of independent allocative inefficiency parameters from six to three. Writing the constrained vector as $\bar{\theta}$, it remains the case that $[\pi(q) - \pi(q, \bar{\theta})] = 0$ if $\bar{\theta} = 1$, and that $[\pi(q) - \pi(q, \bar{\theta})] \geqq 0$ if any $\bar{\theta}_{ij} \neq 1$, again by virtue of the convexity property π.4. The only real effect of forcing allocative inefficiency to be consistent is to blur the distinctions among output mix, input mix, and scale types of allocative inefficiency. It can be shown that (a) $\bar{\theta}_{12} \neq 1$ implies both output mix and scale inefficiency, since consistency requires that $\bar{\theta}_{12} \neq 1$ implies at least one of $(\bar{\theta}_{13}, \bar{\theta}_{14}, \bar{\theta}_{23}, \bar{\theta}_{24})$ be non-unitary; (b) for the same reason, $\bar{\theta}_{34} \neq 1$ implies both input mix and scale inefficiency; and (c) again for the same reason, $(\bar{\theta}_{13}, \bar{\theta}_{14}, \bar{\theta}_{23}, \bar{\theta}_{24}) \neq (1, 1, 1, 1)$ implies scale, output mix and input mix inefficiency unless $\bar{\theta}_{13} = \bar{\theta}_{14} = \bar{\theta}_{23} = \bar{\theta}_{24} \neq 1$, in which case it implies only scale inefficiency.

V. Estimation

The system of output supply and input demand equations developed in section III can best be estimated by seemingly unrelated nonlinear regressions (Gallant (1975)). Letting G be the number of equations, the system can be written as

$$y_{ti} = \alpha_i + f_i(x_{ti};\, \theta_i) + \epsilon_{ti}, \quad i = 1, \ldots, G,$$

where x_{ti} are $k_i \times 1$ vectors, θ_i are $p_i \times 1$ vectors, and α_i are scalars. The G-variate errors $(\epsilon_{t1}, \ldots, \epsilon_{tG})'$ are assumed to be independent, to have the same distribution function with mean $\phi_i (i = 1, \ldots, G)$ and positive definite covariance matrix Σ, and to always be of the same sign. The standard interpretation of the equation is that the systematic part gives the "optimum" value of y_{ti} and the nonsystematic part arises out of random shocks identifying technical inefficiencies of one sort or another (see, e.g., Aigner and Chu (1968)). These technical inefficiencies correspond to an underproduction of outputs and an overutilization of inputs, and cause profit to be less than maximum. Thus disturbances are all of the same sign, in this case nonpositive. The equations' functional forms are so configured that every equation contains a constant term α_i. Assumptions concerning the limiting behavior of the other arguments x_{ti} can be found in Jennrich (1969), Malinvaud (1970), and Gallant (1975). Following Gallant (1975) we can then write each equation as

$$y_i = \alpha_i + f_i(\theta) + \epsilon_i, \qquad i = 1, \ldots, G,$$

where

$$y_i = [y_{1i}, y_{2i}, \ldots, y_{Ti}]',$$

$$f_i(\theta_i) = [f_i(x_{1i};\, \theta_i), f_i(x_{2i};\, \theta_i), \ldots, f_i(x_{Ti};\, \theta_i)]',$$

$$\epsilon_i = (\epsilon_{1i}, \epsilon_{2i}, \ldots, \epsilon_{Ti})'.$$

The complete system is then

$$y = \alpha + f(\theta) + \epsilon,$$

where

$$y = (y'_1, y'_2, \ldots, y'_G)',$$

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_G)',$$

$$f(\theta) = (f'_1(\theta_1), f'_2(\theta_2), \ldots, f'_G(\theta_G))',$$

$$\theta = (\theta'_1, \ldots, \theta'_G),$$

$$\epsilon = (\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_G).$$

Define $\mu = \alpha + \phi$ and let $\delta = (\mu, \theta)$, $g(\delta) = \mu + f(\theta)$. If we specify the covariance matrix for ϵ as $\Sigma \otimes I_T$ then it can be shown that

$$\sqrt{T}(\tilde{\delta} - \delta) \rightarrow N(0, \Omega^{-1}),$$

where

$$\Omega = (1/T)G'(\delta)(\Sigma^{-1} \otimes I)G(\delta),$$

$$G(\delta) = \text{diag}\,[G_1(\delta_1), G_2(\delta_2), \ldots, G_G(\delta_G)],$$

$$G_i(\delta_i) = \partial g_i(\delta_i)/\partial \delta_i,$$

and where $\tilde{\delta}$ is the Aitken-type estimator obtained by minimizing

$$S(\delta) = (1/T)(y - G(\delta))'\Sigma^{-1} \otimes I(y - g(\delta))$$

over $\Delta = X_{i=1}{}^G \delta_i$. As a practical matter Σ is unknown. We thus use a three step procedure. The first step involves consistently estimating each equation separately by nonlinear least squares by minimizing

$$S_i(\delta_i) = (1/T)(y_i - g_i(\delta_i))'(y_i - g_i(\delta_i))$$

over δ_i, $i = 1, \ldots, G$. In the second step the residual vectors are formed by

$$e_i = y_i - g_i(\tilde{\delta}_i), \qquad i = 1, \ldots, G.$$

The third step involves estimation of Σ by

$$\hat{\sigma}_{ij} = (1/T)\, e'_i e_j, \qquad i, j = 1, \ldots, G.$$

Identification of average technical inefficiency from estimates of the μ_i is accomplished in the following fashion. Greene (1980) has shown that, regardless of the distribution of the disturbances, if the x_{ti} are well behaved and the errors are distributed as assumed above, consistent estimates of the intercept terms α_i are given by the order statistics of the consistent residuals for each equation. In our model the consistent residuals e_i are formed in the first step by estimating each equation by nonlinear least squares. Because the estimating equations are driven from a maximum profit frontier the consistent order statistics are given by $e_{i(1)} = \min\,(e_{it})$. Thus estimated average technical efficiency is given by $\hat{\phi}_i = e_{i(1)}$ and thus $\hat{\alpha}_i = \hat{\mu}_i - \hat{\phi}_i$.

VI. An Illustration

In order to illustrate the workings of the model outlined above, we use a data base for the U.S. economy developed by Christensen and Jorgenson (1969, 1970), which consists of 39 annual observations from 1929 to 1967 on prices and quantities of two outputs (consumption and investment) and two inputs (labor and capital). Although aggregate time series data do not provide an ideal vehicle for testing a model of efficiency, these data do permit an illustration of the kinds of questions that can be asked of the model.

The first model we consider is the model of section III that permits both technical and inconsistent allocative inefficiency, augmented so as to allow for technical change. Technical change is introduced by adding a term of the form $\lambda_i t$, $i = 1, \ldots, 4$, to each of the four equations in the model. Average rates of output augmentation and input diminution are provided by $(\lambda_i/\bar{y}_i)$, $i = 1,2$, and $(-\lambda_i/\bar{x}_i)$, $i = 3,4$, respectively. Estimates of A, λ, ϕ and θ are reported in table 1. The monotonicity (π.2) and convexity (π.4) properties are satisfied at all observations. Technical change dominates the regressions, with implied average rates of output augmentation and input diminution being 3.5%, 3.9%, 2.8% and 2.8%, respectively.[5] Turning to technical inefficiency, the estimated values of the ϕ_i imply 24.3%, 36.7%, 9.7% and 15.7% average rates of technical inefficiency, respectively. These are large numbers, and are no doubt a consequence of the boom-or-bust nature of the data base. In light of the way in which they are constructed, they should be viewed as upper bound estimates.[6] Although tests of significance are not available, their dispersion suggests that technical inefficiency has been non-neutral. As for allocative inefficiency, most components of θ are substantially, but insignificantly, different from unity. The Wald test statistic (Judge et al. (1980, p. 757)) for the null hypothesis of allocative efficiency has a value of 7.01. This is well within a 90% confidence interval for a chi-square distribution with six degrees of freedom. Keeping in mind its marginal significance, the estimated θ matrix nonetheless suggests a divergence between perceived and actual price ratios, the effects of which depend on the structure of technology as represented by the estimated A matrix. Since $\hat{\theta}_{12} > 1$, output mix inefficiency takes the form of overproduction of y_1 (consumption goods) relative to y_2 (investment goods), and since $\hat{\theta}_{34} < 1$, input mix inefficiency takes the form of overutilization of x_2 (capital) relative to x_1 (labor). The results are not so straightforward with respect to scale inefficiency, with $\hat{\theta}_{13} > 1$ suggesting inefficiently large scale and $(\hat{\theta}_{14}, \hat{\theta}_{23}, \hat{\theta}_{24}) < (1, 1, 1)$ suggesting inefficiently small scale.

The consistent allocative inefficiency model of section IV is estimated as a restricted variant of

[5] When technical change is included it dominates the regressions, and although estimates of A and θ are well behaved, they are rarely significant. When technical change is dropped from the model, estimates of A and θ are highly significant but the implied profit function is not convex. This is consistent with the findings of Diewert and Parkan (1979), who used non-parametric tests to conclude that this data base is not consistent with competitive profit maximizing behavior in the absence of an allowance for technical progress.

[6] Like all other so-called "full frontier" estimates of technical inefficiency, these order statistics are extremely sensitive to outliers in the data base. There appear to be four severe outliers in the data base: the first two and the last two observations. Estimated average rates of technical inefficiency based on the remaining 35 observations decline from 24.3% to 12.9% and from 36.7% to 17.8% for the two outputs, and remain virtually unchanged for the two inputs.

TABLE 1.—ESTIMATES FOR THE MODEL OF INCONSISTENT ALLOCATIVE INEFFICIENCY
(t-statistics in parentheses)

(A,λ) =

347.95	−183.72 (−2.46)	−27.58 (−0.14)	−122.91 (−1.58)	7.88 (5.31)
	83.2	20.45 (0.21)	22.64 (0.15)	4.02 (3.46)
		147.98	−145.92 (−2.39)	−6.09 (−6.52)
			115.78	−3.69 (−3.89)

ϕ =

54.51
37.43
21.16
20.83

θ =

1.13 (0.76)	5.78 (0.20)	0.63 (0.61)
	0.25 (0.10)	0.13 (0.03)
		0.66 (0.82)

the inconsistent allocative inefficiency model. The Wald test statistic for the null hypothesis of consistent allocative inefficiency is 2.42, well within any reasonable acceptance region for the chi-square with three degrees of freedom. Estimates of A, λ, ϕ and $\bar{\theta}$ for the consistent inefficiency model, also augmented so as to allow for technical change, are reported in table 2. The monotonicity and convexity properties remain satisfied at all observations. Technical change again dominates the regressions, with implied average rates of output augmentation and input diminution being 3.7%, 4.7%, 2.8% and 2.8%, respectively. Tabled values of the ϕ_i imply average rates of technical inefficiency of 25.2%, 41.3%, 9.9% and 15.2%, respectively.[7] Constraining allocative inefficiency to be consistent changes the direction of output mix inefficiency, and makes all four components of scale inefficiency indicate excessively large scale. The direction of input mix inefficiency remains unchanged.[8]

VII. Summary and Conclusions

In this paper we have developed a model of the multiproduct firm in which technical and allocative inefficiency are incorporated in an econometrically useful way. Technical inefficiency can be nonneutral and allocative inefficiency can be inconsistent or consistent. Inconsistent allocative inefficiency can be broken down into input mix, output mix, and scale components, although this decomposition is not generally possible under consistent allocative inefficiency. The model also permits the calculation of the cost, in terms of forgone profit, of each of the four components of total inefficiency. Finally, both full and partial allocative efficiency, as well as consistent allocative inefficiency, are testable restrictions on the general model of technical and inconsistent allocative inefficiency.

We have tested a Generalized Leontief specification of the model on a data base for the U.S. economy over the period 1929–1967, using two outputs and two inputs. For the inconsistent inefficiency model we find substantial nonneutral technical inefficiency, and substantial output mix, input mix, and scale inefficiency, each of which is costly. However, neither null hypothesis of allocative efficiency nor consistent allocative inefficiency can be rejected at conventional levels of confidence.

We conclude by mentioning two factors that we have not incorporated into the model, and that may have influenced our empirical results. First, we have assumed markets to be competitive. However, the scale inefficiency we have found might actually represent a divergence between output prices and marginal revenues which presumably would result from monopoly pricing in the face of scale economies. Thus the

[7] Eliminating the four outliers cited above in note 6 reduces estimated average rates of technical inefficiency associated with the two outputs from 25.5% to 12.4% and from 41.3% to 17.6%, respectively.

[8] Annual values of forgone profit associated with technical and each type of allocative inefficiency for both models are available from the authors.

TABLE 2.—ESTIMATES FOR THE MODEL OF CONSISTENT ALLOCATIVE INEFFICIENCY
(t-statistics in parentheses)

(A,λ) =	347.53	−172.50 (−2.29)	−49.90 (−0.45)	−119.31 (−1.79)	8.24 (5.55)
		84.24	−16.76 (−0.18)	49.84 (0.83)	4.76 (4.09)
			135.82	−135.22 (−2.60)	−6.16 (−6.60)
				108.18	−3.70 (−3.90)

ϕ =	56.61
	42.12
	21.64
	20.23

$\bar{\theta}$ =	0.69 (0.55)	1.43 (0.31)	1.33 (1.00)
		2.08 (0.13)	1.93 (0.67)
			0.93 (1.01)

interpretation of scale inefficiencies could be changed to a measure of monopoly power in product markets.[9] Second, we have assumed that all prices are known with certainty. However, it is well known (see, e.g., Perrakis (1980) for a recent treatment) that price uncertainty can distort quantity choice in much the same way as inefficiency can.

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[9] We are indebted to Robert Pollak for this suggestion.

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