

A Novel Method for Panel Models with Jump Discontinuities in the Parameters: An Application to the Impact of Algorithmic Trading on Financial Market Quality

Bada O.*¹, Gualtieri J. ^{†2}, Kneip A. ^{‡1} and Sickles R. C.^{§2}

¹Institute for Financial Economics and Statistics, University of Bonn

²Department of Economics, Rice University

This version: December 3, 2015

Abstract

While a substantial literature on structural break change point analysis exists for univariate time series, research on large panel data models has not been as extensive. In this paper, a novel method for estimating panel models with multiple structural changes is proposed. The breaks are allowed to occur at unknown points in time and may affect the multivariate slope parameters individually. Our method is related to the Haar wavelet technique; we adjust it according to the structure of the observed variables in order to detect the change points of the parameters consistently. We also develop methods to address endogenous regressors within our modeling framework. The asymptotic property of our estimator is established. In our application, we examine the impact of algorithmic trading on standard measures of market quality such as liquidity and volatility over a time period that covers the financial meltdown that began in 2007. We are able to detect jumps in regression slope parameters automatically without using ad-hoc subsample selection criteria.

*oualid.bada@commerzbank.de

[†]James.gualtieri@ey.com

[‡]akneip@uni-bonn.de

[§]rsickles@rice.edu

1 Introduction

Panel datasets with large cross-sectional dimensions and large numbers of time observations are becoming increasingly available due to the impressive progress of information technology. Parallel progress has also occurred in the econometric literature, by the development of new methods and techniques for analyzing large panels. An important issue that has been addressed in several contexts is the risk of neglecting structural breaks in the data generating process, especially when the observation period is large. Our paper contributes to this literature by providing a general framework to estimate panel models with multiple structural changes that occur at unknown points in time and may affect the model parameters individually. We also develop methods to address endogeneous regressors within our modeling framework.

Given a dependent variable Y_{it} observed for $i = 1, \dots, n$ individuals at $t = 1, \dots, T$ time points, we consider the model

$$Y_{it} = \mu + \sum_{p=1}^P \sum_{j=1}^{S_p+1} X_{it,p} \mathbf{I}(\tau_{j-1,p} < t \leq \tau_{j,p}) \beta_{\tau_{j,p}} + \alpha_i + \theta_t + \varepsilon_{it}, \quad (1)$$

where $\mathbf{I}(\cdot)$ is the indicator function, $X_{it,p}$, $p = 1, \dots, P$, are explanatory variables, α_i is an individual specific effect, θ_t is a common time parameter, and ε_{it} is an unobserved idiosyncratic term that may be correlated with one or more explanatory variables. For each variable $X_{it,p}$, $p = 1, \dots, P$, the corresponding slope parameter is piecewise constant with an unknown set of jump points $\{\tau_{0,p}, \tau_{1,p}, \dots, \tau_{S_p+1,p} \mid \tau_{0,p} = 1 < \tau_{1,p} < \dots < \tau_{S_p+1,p} = T\} \subseteq \{1, \dots, T\}$ for some unknown $S_p \geq 1$.

In single time series, the available information is often not sufficient to uncover the true dates of the structural breaks. Only the time fractions of the break locations can be consistently estimated and tested; see, e.g., Aït-Sahalia and Jacod (2009), Bai (1997), Bai and Perron (1998, 2003), Carr and Wu (2003), Li and Perron (2015), and Pesaran et al. (2011). In panel data models, such a limitation can be alleviated since the cross-section dimension provides an important source of additional information. Besides the virtue of improved statistical efficiency, the determination of the change point locations can be of particular importance in many applications. Indeed, estimating the number and locations of the structural breaks alleviates concerns about ad-hoc subsample selection, enables interpretation of historical events that are not explicitly considered in the model, and avoids related issues of statistical under- or over-parametrization.

One of the earliest contributions to the literature on testing for structural breaks in panel data is the work of Han and Park (1989). The authors propose a multivariate version of the consum-test, which can be seen as a direct extension of the univariate time series test proposed by Brown et al. (1975). Qu and Perron (2007) extend the work of Bai and Perron (2003) and

consider the problem of estimating, computing, and testing multiple structural changes that occur at unknown dates in linear multivariate regression models. They propose a quasi-maximum likelihood method and likelihood ratio-type statistics based on Gaussian errors. The method, however, requires that the number of equations be fixed and does not consider the case of large panel models with unobserved effects and possible endogenous regressors. Based on the work of Andrews (1993), De Wachter and Tzavalis (2012) propose a break testing procedure for dynamic panel data models with exogenous or pre-determined regressors when n is large and T is fixed. The method can be used to test for the presence of a structural break in the slope parameters and/or in the unobserved fixed effects. However, their assumptions allow only for the presence of a single break. Bai (2010) proposes a framework to estimate the break in means and variances. He also considers the case of one break and establishes consistency for both large and fixed T . Kim (2014) extends the work of Bai (2010) to allow for the presence of unobserved common factors in the model. Pauwels et al. (2012) analyze the cases of a known and an unknown break date and propose a Chow-type test allowing for the break to affect some, but not all, cross-section units. Although the method concerns the one-break case, it requires intensive computation to select the most likely individual breaks from all possible sub-intervals when the break date is unknown. Qian and Su (2014) propose a three step procedure. In a first step they use a "naïve" estimator of the slope parameter based on fits over the n individuals for every single time point t . This is used to construct appropriate weights for the adaptive Lasso procedure used in the second step. In a third step they then propose a post Lasso procedure. An important point is that their theory requires that the preliminary naïve estimators of the slope parameters are consistent (otherwise the weights in the adaptive Lasso procedure may be completely irregular). This of course will only work for large n . Their method does not consider cases in which n is small and T is large.

Our work deals with the problem of multiple jump discontinuities in the parameters of panel models without imposing restrictive assumptions on the number, the location, and/or the aspect of the breaks. Our approach is quite general and covers the most important situations encountered in panel data analysis. The method can be applied to panel data with large time span T and large cross-section dimension n and allows for T to be very long compared to n . We also consider the classic case of panel data, in which T is fixed and only n is large. The special structure of panel data as well as the fact that the cross-section dimension n may be arbitrarily large makes it difficult to justify a reliance on methods developed in time series analysis (for fixed n). This also holds for tests proposed in this context¹.

¹Moreover, much of the time series literature on break points is based on the assumption that the number of break points S is known (Li and Perron (2015) would appear to

First of all any asymptotics $T \rightarrow \infty$ for fixed n is only interpretable if the cross-section dimension n is quite small compared to T . If n is large, the corresponding asymptotic expansion may not provide suitable approximations. When applying time series approaches another problem is the special structure of panel data. Due to individual effects, error terms for identical individuals are highly correlated and do not follow the usual assumptions made in the time series context. A model naively based on differences does not fit either, since then the jumping parameters $\beta_{\tau_j, p}$ and $\beta_{\tau_j-1, p}$, $p = 1, \dots, P$, simultaneously occur in the equation.

Our theoretical construct considers breaks in a two-way panel data model, in which the unobserved heterogeneity is composed of additive individual effects and time specific effects. We show that our method can also be extended to cover the case of panel models with unobserved heterogeneous common factors as proposed by Ahn et al. (2001), Pesaran (2006), Bai (2009), Kneip et al. (2012), and Bada and Kneip (2014). We are unaware of other work that provides such a treatment at this level of generality. Our estimation procedure is related to the Haar wavelet technique, which we transform and adapt to the structure of the observed variables in order to detect the location of the break points consistently. We propose a general setup allowing for endogenous models such as dynamic panel models and/or structural models with simultaneous panel equations. Consistency under weak forms of dependency and heteroskedasticity in the idiosyncratic errors is established and the convergence rate of our slope estimator is derived. To consistently detect the jump locations and test for the statistical significance of the breaks, we propose post-wavelet procedures. We prove that our final estimator of the model parameters have the same asymptotic distribution as the (infeasible) estimators that would be obtained if all jump locations were exactly known a priori and thus possess the "oracle property". Our simulations show that, in many configurations of the data, our method performs very well even when the idiosyncratic errors are affected by weak forms of serial-autocorrelation and/or heteroskedasticity.

Although our approach has some similarity to the likelihood based approach of Li and Perron (2015), it is not clear how their approach could be implemented in a panel data context. That appears not to be the case with the approach of Bai and Perron (1998, 2003), which could in principle also be used for panel data. Indeed Bai and Perron are instructive in that they can help to motivate our basic ideas. For a given number S of breakpoints, the methods of Bai and Perron are based on comparing the local fits of each

be an exception). For given S the papers in this literature then propose an estimation procedure, derive asymptotic results, and suggest suitable test procedures. An "estimate" of S relies on sequential testing procedures. In contrast, our procedure does not require any specification of S . It automatically provides an estimate \hat{S} , and we are able to show that with probability tending to 1 our procedure will (asymptotically) provide consistent estimates of all jump locations.

possible combination of $S + 1$ subintervals. Although the method in Bai and Perron (1998) is computationally infeasible, the dynamic programming approach in Bai and Perron (2003) requires $O(T^2)$ fits of different subintervals. Our approach is also based on local fitting of subintervals, using Haar wavelet ideas, which only needs to determine local fits for T subintervals. Another important advantage of our method over the methods of Bai and Perron (1998, 2003), Qu and Perron (2007), and Qian and Su (2014) is that the elements of the slope parameter vector are not forced to jump simultaneously. Our method allows for the total number of jumps to be the sum of S_p individual parameter jumps at some unknown $\tau_{1,p}, \dots, \tau_{S_p,p}$ that are estimated and identified individually for each regressor $p = 1, \dots, P$. Because the method of Bai and Perron (1998, 2003) is based on optimizing the objective function over all possible sub-interval combinations, allowing for the P-slope parameters to jump individually will dramatically increase the number of fits in their algorithm (exponentially on the number of regressors).

Our empirical vehicle for highlighting this new methodology addresses the stability of the relationship between Algorithmic Trading (AT) and Market Quality (MQ). We propose to automatically detect jumps in regression slope parameters to examine the effect of algorithmic trading on market quality in different market situations. We find evidence that the relationship between AT and MQ was disrupted between 2007 and 2008. This period coincides with the beginning of the subprime crisis in the US market and the bankruptcy of the big financial services firm Lehman Brothers and our findings have important implications for proponents and critics of high-frequency trading.

The remainder of the paper is organized as follows. Section 2 explains the basic idea of our estimation procedure by using a relatively straightforward centered univariate panel model. In Section 3, we consider panel models with unobserved effects and multiple jumping slope parameters, present our model assumptions, and derive the main asymptotic results. Section 4 proposes a post-wavelet procedure to estimate the jump locations, derives the asymptotic distribution of the final estimator, and describes selective testing procedures. In Section 5, we discuss models with an issue of omitted common factors and endogenous models arising from structural simultaneous equation systems. Section 6 presents the simulation results of our Monte Carlo experiments. Section 7 focuses on the empirical application. The conclusion follows in Section 8. The mathematical proofs are collected in Appendix A.

2 Basic Concepts

A Simple Panel Model with one Jumping Parameter

To simplify exposition of our basic approach, we begin with a special case of model (1). We consider a centered univariate panel data model of the form

$$Y_{it} = X_{it}\beta_t + e_{it} \quad \text{for } i \in \{1, \dots, n\} \text{ and } t \in \{1, \dots, T\}, \quad (2)$$

where X_{it} is a univariate regressor, $E(e_{it}) = 0$, and β_t is a scalar with

$$\beta_t = \sum_{j=1}^{S+1} \mathbf{I}(\tau_{j-1} < t \leq \tau_j) \beta_{\tau_j}, \quad (3)$$

for some $\tau_0 = 1 < \tau_1 < \dots < \tau_{S+1} = T$, where $S \geq 1$, τ_1, \dots, τ_S , as well as the coefficients β_{τ_j} , $j = 1, \dots, S+1$, are unknown.

Some Fundamental Concepts of Wavelet Transform

The idea behind our approach basically consists of using the Haar wavelet expansion of β_t to control for its piecewise changing character. Before continuing with the estimation method, we introduce some important concepts and notations that are necessary for our analysis.

We assume that the intertemporal sample size T is *dyadic*, i.e., $T = 2^{L-1}$ for some positive integer $L \geq 2$. This is a technical assumption that is only required for constructing the wavelet basis, which are defined in $[1, T]$ via different time dilations of order 2^l , for $l \in \{1, \dots, L\}$. In practice, such an assumption does not impose any real restriction, since one can replicate the data by reflecting the observations at the boundaries until getting the desired dimension property. If, for instance, $T = 125$, we can extend the sample $(Y_{i1}, X_{i1}), \dots, (Y_{iT}, X_{iT})$ with the three last observations (Y_{iT}, X_{iT}) , (Y_{iT-1}, X_{iT-1}) , and (Y_{iT-2}, X_{iT-2}) for $T+1, T+2$, and $T+3$, respectively, to get a new intertemporal size of $2^7 = 128$.

Technically, the discrete wavelet expansion is much like the discrete Fourier transformation, except that the wavelet expansion is constructed with a two parameter system: a dilation level $l \in \{1, \dots, L\}$ and a translation index $k \leq 2^{l-2}$. Let $\{\varphi_{l_0,k}, k = 1, \dots, K_{l_0}\}$, and $\{\psi_{l,k}, l = l_0 + 1, \dots, L; k = 1, \dots, 2^{l-2}\}$, respectively, represent collections of discrete scaling and wavelet functions defined on the discrete interval $\{1, \dots, 2^{L-1}\}$ such that

$$\psi_{l,k}(t) = a_l^\psi I_{l,2k-1}(t) - a_l^\psi I_{l,2k}(t) \quad \text{and} \quad (4)$$

$$\varphi_{l_0,k}(t) = a_{l_0}^\varphi I_{l_0+1,2k-1}(t) + a_{l_0}^\varphi I_{l_0+1,2k}(t), \quad (5)$$

where $a_{l_0}^\varphi = \sqrt{2^{l_0-1}}$, $a_l^\psi = \sqrt{2^{l-2}}$, and $I_{l,m}(t)$ is the indicator function that carries the value one if $t \in \{2^{L-l}(m-1)+1, \dots, 2^{L-l}m\}$ and zero otherwise.

Then the multiscale discrete Haar wavelet expansion of β_t can be presented as follows:

$$\beta_t = \sum_{k=1}^{K_{l_0}} \varphi_{l_0,k}(t) d_{l_0,k} + \sum_{l=l_0+1}^L \sum_{k=1}^{K_l} \psi_{l,k}(t) c_{l,k}, \quad \text{for } t \in \{1, \dots, T\}, \quad (6)$$

where $K_l = 2^{l-2}$, for $l > 1$, and $K_1 = 1$. The coefficients $d_{l,k}$ and $c_{l,k}$ are called scaling and wavelet coefficients, respectively. Because $\varphi_{l_0,k}(t)$ and $\psi_{l,k}(t)$ are orthonormal, $d_{l,k}$ and $c_{l,k}$ are unique and can be interpreted as the projection of β_t on their corresponding bases, i.e., $d_{l_0,k} = \frac{1}{2^{L-1}} \sum_{t=1}^{2^{L-1}} \varphi_{l_0,k}(t) \beta_t$ and $c_{l,k} = \frac{1}{2^{L-1}} \sum_{t=1}^{2^{L-1}} \psi_{l,k}(t) \beta_t$.

Although the Haar wavelet basis functions are the simplest basis within the family of wavelet transforms, they have properties that facilitate the analysis of functions with sudden piecewise changes, as we discuss below.

Orthonormalization and Estimation

Note that the collection of functions, in (6), is not unique. Here, we set $l_0 = 1$, to fix the primary scale to be the coarsest possible with only one parameter that reflects the general mean of β_t . In addition, we propose a slightly modified version of wavelet expansion to adapt the orthonormalization conditions to the requirements of our panel data method.

We consider the following expansion:

$$\beta_t = \sum_{l=1}^L \sum_{k=1}^{K_l} w_{l,k}(t) b_{l,k} \quad \text{for } t \in \{1, \dots, T\}, \quad (7)$$

where

$$w_{l,k}(t) = \begin{cases} a_{1,1} = a_{2,1} h_{2,1}(t) + a_{2,2} h_{2,2}(t) & \text{if } l = 1, \text{ and} \\ a_{l,2k-1} h_{l,2k-1}(t) - a_{l,2k} h_{l,2k}(t) & \text{if } l > 1, \end{cases} \quad (8)$$

for some positive standardizing scales $a_{l,2k-1}$ and $a_{l,2k}$ that, unlike the conventional wavelets, depend on both the dilation level l and translation index k . Their exact form will be discussed in detail below. We define the function $h_{l,m}(t)$ as follows:

$$h_{l,m}(t) = \sqrt{2^{l-2}} I_{l,m}(t). \quad (9)$$

The most appealing feature of the expansion (7) (and (6) with $l_0 = 1$) is that the set of the wavelet coefficients $\{b_{l,k}\}$ contains at most $(S + 1)L$ non-zero-wavelet coefficients. This important property results from the fact that each jump in β_t can be sensed at each dilation level by at most one translation function. Proposition 1 states the existence of (7) for any arbitrary positive real scales $a_{l,2k}$ and $a_{l,2k-1}$.

Proposition 1 Suppose $T = 2^{L-1}$, for some integer $L \geq 2$, and $\beta = (\beta_1, \dots, \beta_T)' \in \mathbb{R}^T$ a vector that possesses exactly $S \geq 1$ jumps at $\{\tau_1, \dots, \tau_S | \tau_1 < \dots < \tau_S\} \subseteq \{1, \dots, T\}$ as in (3). Let $a_{1,1}, a_{l,2k-1}$ and $a_{l,2k}$ be arbitrary positive real values for all $l \in \{1, \dots, L\}$, and $k \in \{1, \dots, K_l\}$. Thus, expansion (7) exists and the set of the wavelet coefficients $\{b_{lk} | l = 1, \dots, L; k = 1, \dots, K_l\}$ contains at most $(S+1)L$ non-zero coefficients.

Remark 1 If we try to naively use a dummy for each time point to construct an extension with exactly $(S+1)$ non-zero coefficients, we end up with highly correlated basis vectors and the differentiation between the unknown zero and non-zero coefficients in the presence of noise becomes quite problematic. The Haar wavelet basis functions can be seen as a special design of dummy vectors that has many technical and computational advantages. We show below that our structure adapted wavelet estimator achieves mean square consistency even if the cross-section dimension n is fixed. This is not possible when naively using T dummy vectors.

Using (7), we can rewrite Model (2) as

$$Y_{it} = \sum_{l=1}^L \sum_{k=1}^{K_l} \mathcal{X}_{l,k,it} b_{l,k} + e_{it}, \quad (10)$$

where

$$\mathcal{X}_{l,k,it} = X_{it} w_{l,k}(t).$$

In vector notation,

$$Y_{it} = \mathcal{X}'_{it} b + e_{it}, \quad (11)$$

where $\mathcal{X}_{it} = (\mathcal{X}_{1,1,it}, \dots, \mathcal{X}_{L,K_L,it})'$ and $b = (b_{1,1}, \dots, b_{L,K_L})'$.

To capture the structural breaks of β_t , we propose, in a first step, to estimate the vector of the wavelet coefficients b in (11). Throughout, we assume the existence of an instrument Z_{it} that is correlated with X_{it} and fulfills $E(Z_{it}e_{it}) = 0$ for all i and t . The idea behind this assumption is to provide a general treatment that allows for estimating models with endogenous regressors such as dynamic models or structural models with simultaneous equations. Let $\mathcal{Z}_{l,k,it} = Z_{it} w_{l,k}(t)$ and $\mathcal{Z}_{it} = (\mathcal{Z}_{1,1,it}, \dots, \mathcal{Z}_{L,K_L,it})'$. Because $E(Z_{it}e_{it}) = 0$ for all i and t , we can infer that $E(\mathcal{Z}_{l,k,it}e_{it}) = 0$, for all l and k . The required theoretical moment condition for estimating b is

$$E(\mathcal{Z}_{it}(Y_{it} - \mathcal{X}'_{it}b)) = 0. \quad (12)$$

The IV estimator of b (hereafter, denoted by \tilde{b}) is obtained by solving the empirical counterpart of (12), i.e.,

$$\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (\mathcal{Z}_{it}(Y_{it} - \mathcal{X}'_{it}\tilde{b})) = 0. \quad (13)$$

Remark 2 We know from the Generalized Method of Moments (GMM) that the IV estimator is equivalent to the just-identified GMM estimator, in which the number of instruments is equal to the number of parameters to be estimated. Hence, our estimator of b can be seen as the GMM estimator:

$$\tilde{b} = \arg \min_b \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (Y_{it} - \mathcal{X}'_{it}b) \mathcal{Z}'_{it} \mathcal{W}_T \mathcal{Z}_{it} (Y_{it} - \mathcal{X}'_{it}b), \quad (14)$$

where \mathcal{W}_T is an arbitrary symmetric $(T \times T)$ full rank matrix. Since the choice of \mathcal{W}_T in the just-identified case is irrelevant, we can use the identity matrix to solve (14).

Under general assumptions, we can state the consistency of \tilde{b} for any arbitrary collection of wavelet functions. However, the problem with naively using the conventional basis functions is that the identification of the zero- and non-zero coefficients will be ambiguous. Not only will the presence of the error term in (10) affect the estimates of $b_{l,k}$ but also the non-orthogonality of $\mathcal{Z}_{l',k',it}$ to $\mathcal{X}_{l,k,it}$ across different dilation and translation levels in the objective function (the IV moment condition) will move the problem from a classical wavelets shrinkage scheme to a complex model selection problem.

Our solution consists of adjusting the scales $a_{1,1}$, $a_{l,2k-1}$ and $a_{l,2k}$ in (8) to the structure of X_{it} and Z_{it} so that following normalization conditions are satisfied.

- (a): $\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \mathcal{Z}_{l,k,it} \mathcal{X}'_{l',k',it} = 1$ if $(l,k) = (l',k')$ and
- (b): $\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \mathcal{Z}_{l,k,it} \mathcal{X}'_{l',k',it} = 0$ for all $(l,k) \neq (l',k')$.

Proposition 3, in Appendix A.1, gives the mathematical conditions for $a_{1,1}$, $a_{l,2k-1}$ and $a_{l,2k}$ to ensure (a) and (b). The solution is

$$\begin{aligned} a_{1,1} &= Q_{1,1}^{-\frac{1}{2}}, \\ a_{l,2k-1} &= Q_{l,2k-1}^{-1} (Q_{l,2k-1}^{-1} + Q_{l,2k}^{-1})^{-\frac{1}{2}}, \text{ and} \\ a_{l,2k} &= Q_{l,2k}^{-1} (Q_{l,2k-1}^{-1} + Q_{l,2k}^{-1})^{-\frac{1}{2}}, \end{aligned}$$

where $Q_{1,1} = \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T X_{it} Z_{it}$, $Q_{l,2k-1} = \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T X_{it} Z_{it} h_{l,2k-1}^2(t)$, and $Q_{l,2k} = \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T X_{it} Z_{it} h_{l,2k}^2(t)$.

Solving (13) (or (14)) with respect to $b_{l,k}$ under Restrictions (a) and (b), we obtain

$$\tilde{b}_{l,k} = \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \mathcal{Z}_{l,k,it} Y_{it}. \quad (15)$$

Making use of orthonormality, we can directly perform the universal thresholding scheme of Donoho and Johnstone (1994). Our structure adapted wavelet estimator of β_t (hereafter, the SAW estimator) can be obtained by

$$\tilde{\beta}_t = \sum_{l=1}^L \sum_{k=1}^{K_l} w_{l,k}(t) \hat{b}_{l,k}, \quad (16)$$

where

$$\hat{b}_{l,k} = \begin{cases} \tilde{b}_{l,k} & \text{if } |\tilde{b}_{l,k}| > \lambda_{n,T} \text{ and} \\ 0 & \text{else,} \end{cases} \quad (17)$$

for some threshold $\lambda_{n,T}$ that depends on n and T . Theorems 1, 2 give the necessary conditions for $\lambda_{n,T}$ to ensure consistency under Assumptions A-C presented in Section 3.

To detect all possible structural changes, we hence only need T fits, while e.g. Bai and Perron (2003) need $O(T^2)$ subinterval fits to do a similar job.

Remark 3 *If the explanatory variable X_{it} is exogenous, we can choose $Z_{it} = X_{it}$ to instrument all elements in $\mathcal{X}_{l,k,it}$ with themselves. In this case, our shrinkage estimator $\hat{b}_{l,k}$ can be interpreted as a Lasso estimator with the advantage of perfect orthogonal regressors; see, e.g., Tibshirani (1996). More generally, if X_{it} is allowed to be endogenous and $Z_{l,k,it} \neq \mathcal{X}_{l,k,it}$, $\hat{b}_{l,k}$ can be obtained by minimizing a Lasso-penalized just-identified GMM objective function. That is,*

$$\hat{b} = \arg \min_b \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (Y_{it} - \mathcal{X}'_{it} b) \mathcal{Z}'_{it} \mathcal{W}_T \mathcal{Z}_{it} (Y_{it} - \mathcal{X}'_{it} b) + \lambda_{n,T} |b|, \quad (18)$$

where $|b| = \sum_{l=1}^L \sum_{k=1}^{K_l} |b_{l,k}|$ and \mathcal{W}_T is an arbitrary symmetric $(T \times T)$ full rank matrix. Note that (18) and (17) lead to the same result independent of the choice of \mathcal{W}_T .

Now that we have developed procedures for first step SAW estimation for a straightforward centered panel model we turn to generalizations for multivariate models with unobserved heterogeneity effects as well as corresponding post-SAW procedures.

3 Two-way Panel Models with Multiple Jumps

3.1 Model

One of the main advantages of using panel datasets is the possibility of dealing with problems related to the potential effect of unobserved heterogeneity in time- and cross-section dimensions. In this section, we extend the univariate SAW method to allow for multiple jumping parameters in

panel data models with time constant heterogeneity effects and unobserved homogeneous time factor.

Collecting the slope parameters in a $(P \times 1)$ time-varying vector, we can rewrite Model (1) as

$$Y_{it} = \mu + X'_{it}\beta_t + \alpha_i + \theta_t + e_{it}, \quad (19)$$

where $X_{it} = (X_{1,it}, \dots, X_{P,it})'$ is the $(P \times 1)$ vector of regressors, $\beta_t = (\beta_{t,1}, \dots, \beta_{t,P})'$ is a $(P \times 1)$ unknown vector of slope parameters, and for each $\beta_{t,p}$, $p \in \{1, \dots, P\}$, we have

$$\beta_{t,p} = \sum_{j=1}^{S_p+1} \mathbf{I}(\tau_{j-1,p} < t \leq \tau_{j,p}) \beta_{\tau_{j,p}}. \quad (20)$$

Equation (20) allows for each slope parameter $\beta_{t,p}$, $p \in \{1, \dots, P\}$, to change at p -specific unknown break points in the time.

Even in the absence of structural breaks, the uniqueness of μ , α_i and θ_t requires some identification conditions. We impose the commonly used restrictions:

$$\begin{aligned} \text{C.1: } \sum_{i=1}^n \alpha_i &= 0, \text{ and} \\ \text{C.2: } \sum_{t=1}^T \theta_t &= 0. \end{aligned} \quad (21)$$

We want to emphasize that the choice of C.1 and C.2 becomes irrelevant when the focus lies upon estimating the slope parameters $\beta_{t,p}$, $p \in \{1, \dots, P\}$.

3.2 Estimation

The estimation of Model (19) encounters two additional complications compared to the univariate model discussed in Section 2: besides the need to control for the presence of the unknown parameters μ , α_i , and θ_t , we have to deal with multivariate wavelets.

In order to cover the case of dynamic models with both small and large T , we conventionally start with differencing the model to eliminate the individual effects and assume the existence of appropriate instruments. By taking the difference on the left and the right hand side of (19), we have the expression

$$\Delta Y_{it} = X'_{it}\beta_t - X'_{it-1}\beta_{t-1} + \Delta\theta_t + \Delta e_{it}, \quad (22)$$

for $i \in \{1, \dots, n\}$ and $t \in \{2, \dots, T\}$, where Δ denotes the difference operator of first order.

We can eliminate the term $\Delta\theta_t = \theta_t - \theta_{t-1}$ by using the classical within transformation on the model, i.e., transforming ΔY_{it} to $\Delta \dot{Y}_{it} = \Delta Y_{it} - \frac{1}{n} \sum_{i=1}^n \Delta Y_{it}$. Alternatively, we can associate $\Delta\theta_t$ with an additional unit regressor in the model and estimate it jointly with β_t as a potential jumping parameter. Indeed, allowing for $\Delta\theta_t$ to be piecewise constant over time can

be very useful for interpretation, especially when the original time effect θ_t has approximately a piecewise changing linear trend.

Let $\underline{X}_{it} = (X'_{it}, -X'_{it-1}, 1)'$ and $\gamma_t = (\beta'_t, \beta'_{t-1}, \Delta\theta_t)'$ be $(\underline{P} \times 1)$ vectors, where $\underline{P} = 2P + 1$. We can rewrite Model (22) as

$$\begin{aligned}\Delta Y_{it} &= (X'_{it}, -X'_{it-1}, 1) \begin{pmatrix} \beta_t \\ \beta_{t-1} \\ \Delta\theta_t \end{pmatrix} + \Delta e_{it}, \\ &= \underline{X}'_{it} \gamma_t + \Delta e_{it},\end{aligned}\tag{23}$$

for $i \in \{1, \dots, n\}$ and $t \in \{2, \dots, T\}$.

Once the unobserved individual effects are eliminated, the new vector of the slope parameters γ_t can be estimated by using the multivariate version of the structure adapted wavelet method (SAW) introduced in Section 2. The multivariate wavelet expansion of γ_t can be written as:

$$\gamma_t = \sum_{l=1}^L \sum_{k=1}^{K_l} W_{lk}(t) \underline{b}_{l,k} \quad \text{for } t \in \{2, \dots, T\},\tag{24}$$

where L and K_l are defined in the same way as in Section 2, $\underline{b}_{l,k} = (b_{l,k,1}, \dots, b_{l,k,P})'$ is a $(\underline{P} \times 1)$ vector of wavelet coefficients and $W_{lk}(t)$ is a $(\underline{P} \times \underline{P})$ multivariate wavelet basis matrix defined as

$$W_{l,k}(t) = \begin{cases} A_{1,1} = A_{2,1}H_{2,1}(t) + A_{2,2}H_{2,2}(t) & \text{if } l = 1, \text{ and} \\ A_{l,2k-1}H_{l,2k-1}(t) - A_{l,2k}H_{l,2k}(t) & \text{if } l > 1, \end{cases}\tag{25}$$

with

$$H_{l,m}(t) = \sqrt{2^{l-2}} I_{l,m}(t-1).$$

Here, $I_{l,m}(\cdot)$ is the indicator function defined in Section 2, and $A_{1,1}$, $A_{l,2k-1}$, and $A_{l,2k}$ are constructed so that the following orthonormality conditions are fulfilled:

$$(A): \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \underline{Z}_{l,k,it} \underline{X}'_{l',k',it} = I_{\underline{P} \times \underline{P}} \text{ if } (l, k) = (l', k') \text{ and}$$

$$(B): \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \underline{Z}_{l,k,it} \underline{X}'_{l',k',it} = \mathbf{0}_{\underline{P} \times \underline{P}} \text{ for all } (l, k) \neq (l', k'),$$

where $\underline{X}'_{l,k,it} = \underline{X}'_{it} W_{lk}(t)$, $I_{\underline{P} \times \underline{P}}$ is the $(\underline{P} \times \underline{P})$ identity matrix, $\mathbf{0}_{\underline{P} \times \underline{P}}$ is a $(\underline{P} \times \underline{P})$ matrix of zeros, and $\underline{Z}'_{l,k,it} = \underline{Z}'_{it} W_{l,k}(t)$ with \underline{Z}_{it} a $(\underline{P} \times 1)$ vector used to instrument the \underline{P} variables in \underline{X}_{it} ; the unit regressor associated with $\Delta\theta_t$ and the remaining exogenous regressors (if they exist) can be, of course, instrumented by themselves.

We can verify that

$$\begin{aligned}A_{1,1} &= \underline{Q}_{1,1}^{-\frac{1}{2}}, \\ A_{l,2k-1} &= \underline{Q}_{l,2k-1}^{-1} (\underline{Q}_{l,2k-1}^{-1} + \underline{Q}_{l,2k}^{-1})^{-\frac{1}{2}}, \text{ and} \\ A_{l,2k} &= \underline{Q}_{l,2k}^{-1} (\underline{Q}_{l,2k-1}^{-1} + \underline{Q}_{l,2k}^{-1})^{-\frac{1}{2}},\end{aligned}$$

where

$$\begin{aligned}\underline{Q}_{1,1} &= \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \underline{Z}_{it} \underline{X}'_{it}, \\ \underline{Q}_{l,2k-1} &= \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \underline{Z}_{it} \underline{X}'_{it} H_{l,2k-1}(t)^2, \text{ and} \\ \underline{Q}_{l,2k} &= \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \underline{Z}_{it} \underline{X}'_{it} H_{l,2k}(t)^2.\end{aligned}$$

The IV estimator of $\underline{b}_{l,k}$ is the solution to the empirical moment condition

$$\frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \sum_{l=1}^L \sum_{k=1}^{K_l} (\underline{Z}_{l,k,it} (\Delta Y_{it} - \underline{X}'_{l,kit} \tilde{b}_{l,k})) = 0. \quad (26)$$

Solving (26) for $\tilde{b}_{l,k}$ under the the normalization Conditions (A) and (B), we obtain

$$\tilde{b}_{l,k,p} = \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \underline{Z}_{l,k,it,p} \Delta Y_{it},$$

where $\tilde{b}_{l,k,p}$ and $\underline{Z}_{l,k,it,p}$ are the p -th elements of $\tilde{b}_{l,k}$ and $\underline{Z}_{l,k,it}$, respectively.

Let $\lambda_{n,T}$ be a predetermined threshold and $\hat{b}_{l,k,q}$ the estimator of the wavelet coefficients after shrinkage, i.e.,

$$\hat{b}_{l,k,q} = \begin{cases} \tilde{b}_{l,k,q} & \text{if } |\tilde{b}_{l,k,q}| > \lambda_{n,T} \text{ and} \\ 0 & \text{else.} \end{cases} \quad (27)$$

The SAW estimator of the parameters $\gamma_{t,p}$, $p \in \{1, \dots, \underline{P}\}$, composing the vector γ_t can be obtained by

$$\hat{\gamma}_{t,p} = \sum_{l=1}^L \sum_{k=1}^{K_l} \sum_{q=1}^{\underline{P}} W_{lk,p,q}(t) \hat{b}_{l,k,q}, \quad (28)$$

where $W_{lk,p,q}$ is the (p,q) - element of the basis matrix $W_{lk}(t)$, for $p, q \in \{1, \dots, \underline{P}\}$.

Recall that by construction $\gamma_{t,p} = \beta_{t,p}$, $\gamma_{t,p+P} = \beta_{t-1,p}$ for $p \in \{1, \dots, P\}$ and $\gamma_{t,p} = \Delta\theta_t$ for $p = \underline{P}$. A first step estimator of $\beta_{t,p}$ for $t \in \{2, \dots, T\}$ can be obtained by $\hat{\gamma}_{t,p}$. Another natural estimator of $\beta_{t,p}$ for $t \in \{1, \dots, T-1\}$ is $\hat{\gamma}_{t+1,p+P}$. A first step estimator of $\Delta\theta_t$ can be obtained by $\hat{\gamma}_{t,\underline{P}}$. We show, in Section 3.3, the uniform and the mean squared consistency of $\hat{\gamma}_{t,p}$ for all $p \in \{1, \dots, \underline{P}\}$. We propose, however, to use this estimator only as a first step estimator for estimating the jump locations $\tau_{1,p}, \dots, \tau_{S_p,p}$, $p \in \{1, \dots, P\}$. Once these dates are detected, we propose to perform a post-SAW estimation in order to get more efficient estimates and permit classical inferences to be conducted; see Section 4.3.

Remark 4 *The first step estimation of Model (19) is based on the SAW estimation of the transformed Model (22), which is performed without taking into account the fact that $\gamma_{t,p}$ and $\gamma_{t+1,p+P}$ are identical time series of*

parameters. Although the estimators $\hat{\gamma}_{t,p}$ and $\hat{\gamma}_{t+1,p+P}$ are not restricted to be identical for $t \in \{2, \dots, T-1\}$, they present a beneficial property that can be exploited for consistently estimating the jump locations; see Section 4.2. The loss of efficiency due to the extended number of parameters can be redressed through a post-SAW estimation once the jump locations are consistently estimated; see Section 4.3.

3.3 Assumptions and Main Asymptotic Results

We now present a set of assumptions that are necessary for our asymptotic analysis. Throughout, we use $E_c(\cdot)$ to define the conditional expectation given $\{X_{it}\}_{i,t \in \mathbb{N}^{*2}}$ and $\{Z_{it}\}_{i,t \in \mathbb{N}^{*2}}$, where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. We denote by M a finite positive constant, not dependent on n and T . The operators \xrightarrow{p} and \xrightarrow{d} denote the convergence in probability and distribution. $O_p(\cdot)$ and $o_p(\cdot)$ are the usual Landau-symbols. The Frobenius norm of a $(p \times k)$ matrix A is denoted by $\|A\| = [\text{tr}(A' A)]^{1/2}$, where A' denotes the transpose of A .

Our theoretical setup relies on the following assumptions.

Assumption A - Data Dimension and Stability Intervals:

- (i) $T-1 = 2^{L-1}$ for some natural number $L > 1$; the number of regressors P is fixed.
- (ii) $n \rightarrow \infty$; T is either fixed or passes to infinity simultaneously with n such that $\log(T)/n \rightarrow 0$.
- (iii) $\min_{j,p} |\beta_{\tau_{j,p}} - \beta_{\tau_{j-1,p}}|$ does not vanish when n and T pass to infinity; all stability intervals $(\tau_{j,p} - \tau_{j-1,p}) \rightarrow \infty$ uniformly in n , as $T \rightarrow \infty$.

Assumption B - Regressors and Instruments:

- (i) for all i and t , $E_c(\underline{Z}_{it} e_{it}) = 0$; for all $l \in \{1, \dots, L\}$ and $k \in \{1, \dots, K_l\}$,

$$\underline{Q}_{l,k} = \frac{1}{n \cdot \#\{s | h_{l,k}(s) \neq 0\}} \sum_{t \in \{s | h_{l,k}(s) \neq 0\}} \sum_{i=1}^n \underline{Z}_{it} \underline{X}'_{it} \xrightarrow{p} \underline{Q}_{l,k}^\circ,$$

where $\underline{Q}_{l,k}^\circ$ is a $(\underline{P} \times \underline{P})$ full rank finite matrix with distinct eigenvectors.

- (ii) The moments $E\|\underline{Z}_{it}\|^4$ and $E\|\underline{X}_{it}\|^4$ are bounded uniformly in i and t ; for $A_{l,2k} = \underline{Q}_{l,2k}^{-1} (\underline{Q}_{l,2k}^{-1} + \underline{Q}_{l,2k-1}^{-1})^{-1/2}$ and $A_{l,2k-1} = \underline{Q}_{l,2k-1}^{-1} (\underline{Q}_{l,2k}^{-1} + \underline{Q}_{l,2k-1}^{-1})^{-1/2}$, the moments $E\|A_{l,2k}\|^4$ and $E\|A_{l,2k-1}\|^4$ are bounded uniformly in l and k .

- (iii) the multivariate distribution of $\{\Delta e_{it}\}_{i \in \mathbb{N}^*, t \in \mathbb{N}^* \setminus \{1\}}$ is Sub-Gaussian so that every linear combination

$$\Pi_{nT}(a_{s,s'}) = \sum_{t=s+1}^{s'} \sum_{i=1}^n \frac{a_{s,s',it}}{\sqrt{n(s'-s)}} \Delta e_{it},$$

with $E(a_{s,s',it} \Delta e_{it}) = 0$ and $E(\Pi_{nT}^2(a_{s,s'})) \leq M$, is Sub-Gaussian distributed of order $\Sigma_{nT}(a_{s,s'}) = E(\Pi_{nT}^2(a_{s,s'}))$, i.e.,

$$P(\Sigma_{nT}^{-\frac{1}{2}}(a_{s,s'}) |\Pi_{nT}(a_{s,s'})| \geq c) \leq \frac{1}{c} \exp(-\frac{c^2}{2}),$$

for any $c > 0$.

Assumption C - Weak Dependencies and Heteroskedasticity in the Error Term: $E_c(\Delta e_{it} \Delta e_{jm}) = \sigma_{ij,tm}$, $|\sigma_{ij,tm}| \leq \bar{\sigma}$ for all (i, j, t, m) such that

$$\frac{1}{n(s'-s+1)} \sum_{i=1}^n \sum_{j=1}^n \sum_{t=s+1}^{s'} \sum_{m=s+1}^{s'} |\sigma_{ij,tm}| \leq M.$$

Assumption A.(i) specifies a dyadic condition on the intertemporal data size T . This is a technical assumption that is only required for constructing the dyadic wavelet basis functions. As mentioned earlier, in practice, one can replicate the data by reflecting the observations at the boundaries to get the desired dimension. The asymptotic property of the estimator, will, however depend on the original data size and not on the size of the replicated data. Assumption A.(ii) allows for the time dimension T to be very long compared to n but in such a way that $\log(T) = o(n)$. A.(ii) considers also the classical case of panel data, in which T is fixed and only $n \rightarrow \infty$. Assumption A.(iii) guarantees that the jumps do not vanish as n and/or T pass to infinity. The second part of Assumption A.(iii) can be alleviated to allow for some stability intervals to stay fixed if $T \rightarrow \infty$. Assuming the stability intervals to pass to infinity when T gets large allows for interpreting the T -asymptotic as a full-in asymptotic.

Assumption B.(i) requires that the probability limit of $Q_{L,k}$ is a full rank finite matrix with distinct eigenvectors. This is to ensure that its eigen-value decomposition exists. Assumption B.(ii) specifies commonly used moment conditions to allow for some limiting terms to be $O_p(1)$ when using Chebyshev inequality. The Sub-Gaussian condition in Assumption B.(iii) excludes heavy tailed distributed errors but does not impose any specific distribution.

Assumption C allows for a weak form of time series and cross section dependence in the errors as well as heteroskedasticities in both time and cross-section dimension. It implies that the covariances and variances are

uniformly bounded and the double summations over all possible time partitions are well behaved. The assumption generalizes the restricted case of independent and identically distributed errors.

The following Lemma establishes the main asymptotic results for the structure adapted wavelet coefficients.

Lemma 1 *Suppose Assumptions A-C hold, then*

(i)

$$\sup_{l,k,q} \left| \tilde{b}_{l,k,q} - b_{l,k,q} \right| = O_p(\sqrt{\log(T-1)/n(T-1)}),$$

(ii) *for some finite $M > \sqrt{2}$,*

$$\sup_{l,k,q} \left| \tilde{b}_{l,k,q} - b_{l,k,q} \right| \leq M \sqrt{\log((T-1)P)/n(T-1)}$$

holds with a probability that converges to 1 independently of n , as $T \rightarrow \infty$.

Theorem 1 establishes the uniform and the mean square consistency of $\tilde{\gamma}_{t,p}$.

Theorem 1 *Assume Assumptions A-C, then the following statements hold:*

(i) $\sup_{t \in \{2, \dots, T\}} |\hat{\gamma}_{t,p} - \gamma_{t,p}| = o_p(1)$ for all $p \in \{1, \dots, P\}$, if $\sqrt{T-1}\lambda_{n,T} \rightarrow 0$, as $n, T \rightarrow \infty$ or $n \rightarrow \infty$ and T is fixed, and

(ii) $\frac{1}{T-1} \sum_{t=2}^T \|\hat{\gamma}_t - \gamma_t\|^2 = O_p\left(\frac{J^*}{(T-1)} (\log(T-1)/n)^\kappa\right)$, where $J^* = \min\{(\sum_{p=1}^P S_p + 1) \log(T-1), (T-1)\}$, if $\sqrt{T-1}\lambda_{n,T} \sim (\log(T-1)/n)^\kappa/2$, for any $\kappa \in]0, 1[$.

Uniform consistency is obtained when $n \rightarrow \infty$ and T is fixed or $n, T \rightarrow \infty$ with $\log(T)/n \rightarrow 0$. If the maximum number of jumps is fixed, the mean square consistency is obtained even when n is fixed and only $T \rightarrow \infty$.

In our wavelet approach, accuracy of parameter estimates improves with n and T . From Lemma 1 and Theorem 1, we can show that the mean square error of our SAW estimates of $\beta_{t,p}, t \in \{1, \dots, T\}$, converges at a rate of $(\log T)^2/(Tn)$, for some $\lambda_{n,T}$, while the mean square error for the dummy approach discussed in Remark 1 only converges at a rate of $1/n$. The explanation for this effect lies in the special structure of wavelets. It is a basis expansion, where each basis function is local and describes the behavior in a specific subinterval. Only the first basis function is global. The corresponding parameter is fitted over all nT observations. With the fitted first basis function we essentially quantify the best global fit with a constant β . At each resolution level $L = 1, 2, 3, \dots$ we can then analyze parameter changes in $2^{(L-1)}$ local subintervals, which correspond to the

respective basis functions. Each of these subintervals contains approximately $nT/2^{(L-1)}$ observation points. This means that the “relative error” of the coefficient estimates increases as L increases. The coefficients of the lowest level basis functions are only determined from $2n$ observations and are the least accurate as they are highly influenced by noise.

Accuracy of our SAW estimate also depends on whether or not the jump locations are dyadic. As an illustration, consider the case of one jump at some location $\tau = \tau_{1,1} = \dots = \tau_{1,P}$. If $\tau = (T-1)/2$, then only the global fit and another basis function is necessary, and the mean squared error of the parameter estimates is proportional to $1/(nT)$. The same is true if e.g. $\tau = 3(T-1)/4$, in which case one additional resolution level is required, but the mean squared error will still be proportional to $1/(nT)$. Also, with probability tending to 1 the estimated location will converge to the true location even for fixed n . For arbitrary (non-dyadic) locations of the jump points, the inaccurate highest resolution level also will play a role. In this case for fixed n the true point might not be exactly identified even for large T . Nevertheless, the information acquired from higher levels may still allow us to say with high probability that the true point lies, e.g., between $(6/8)T$ and $(7/8)T$. Theorem 1 proves that the distance between the true and estimated location will be at least of order $O_P(\log T/\sqrt{n\kappa})$, i.e., $\sqrt{T}|\hat{\tau}/T - \tau/T| = O_P(\log T/\sqrt{Tn\kappa})$. For fixed n and simultaneous parameter jumps, Bai and Perron (2003) arrive at similar theoretical results and thus our approach may be seen as a computationally more efficient variant when there are only T subinterval fits.

A threshold that satisfies Conditions (i) and (ii) in Theorem 1, can be constructed as follows:

$$\lambda_{nT} = \hat{V}_{nT}^{\frac{1}{2}} \left(\frac{2 \log((T-1)P)}{n(T-1)^{1/\kappa}} \right)^{\kappa/2}, \text{ for some } \kappa \in]0, 1[, \quad (29)$$

where \hat{V}_{nT} is the empirical variance estimator corresponding to the largest variance of $\frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T \mathcal{Z}_{it,l,k,p} \Delta e_{it}$ over l, k , and p . Such an estimator can be obtained by using the residuals \tilde{e}_{it} of a pre-intermediate SAW regression performed with a plug-in threshold $\lambda_{nT}^* = 0$. We want to emphasize that asymptotically all we need is that \hat{V}_{nT} be strictly positive and bounded. The role of $\hat{V}_{nT}^{\frac{1}{2}}$ is only to give the threshold a convenient amplitude. The role of $\kappa < 1$ is to trade off the under-estimation effect that can arise from the plug-in threshold $\lambda_{n,T}^* = 0$. An ad-hoc choice of κ is $1 - \log \log(nT) / \log(nT)$. For more accurate choices, we refer to the calibration strategies proposed by Hallin and Liška (2007) and Alessi et al. (2010).

Remark 5 *So far, we have considered the estimation of jumping slope parameters in the case of stochastically bounded regressors; see Assumption B. Our method should also be appropriate for panel co-integration models. If*

the observed variables are integrated, for instance $I(1)$, the convergence rate in Lemma 1 will be different (in general faster), but the shrinkage idea and the consistency of the jumping slope estimator remain valid. All we need is an appropriate threshold that asymptotically dominates the supremum estimator of the zero coefficients $\sup_{l,k,q|b_{l,k,q}=0} |\tilde{b}_{l,k,q}|$ but is strictly dominated by all non-zero coefficients $\{|b_{l,k,q}| \neq 0\}$ as n and/or T go to infinity.

4 Post-SAW Procedures

4.1 Tree-Structured Representation

The intrinsic problem of wavelets is that wavelet functions are constructed via dyadic dilations. Error may make this feature spuriously generate some additional mini jumps to simulate the big (true) jump when it is located at a non-dyadic position. To construct a selective inference for testing the systematic jumps it is important to encode the coefficients that may generate such effects. One possible approach is to examine the so-called *tree-structured* representation, which is based on the hierarchical interpretation of the wavelet coefficients. Recall that the wavelet basis functions are nested over a binary multiscale structure so that the support of an (l, k) -basis (the time interval in which the basis function is not zero) contains the supports of the basis $(l+1, 2k-1)$ and $(l+1, 2k)$. We say that the wavelet coefficient $b_{l,k}$ is the *parent* of the two *children* $b_{l+1,2k-1}$ and $b_{l+1,2k}$. This induces a dyadic tree structure rooted to the primary parent $b_{1,1}$. To encode the possible systematic jumps, we have to traverse the tree up to the root parent in a recursive trajectory starting from the non-zero coefficients at the finest resolution (highest dilation level). While the presence of a non-zero coefficient, at the highest level, indicates the presence of a jump, the parent may have a non-zero coefficient only to indicate that the stability interval around this jump is larger than its support.

As an illustration, consider the tree-structured representation in Figure 1. The coefficients at the not-ringed nodes fall in the interval $[-\lambda_{n,T}, \lambda_{n,T}]$ and carry the value zero. Starting from the non-zero coefficient $\tilde{b}_{5,6}$ at the finest resolution and traversing the tree up to the root parent, we can identify $\tilde{b}_{4,3}$, $\tilde{b}_{3,2}$, and $\tilde{b}_{2,1}$ as candidates for generating potential visual artifacts at points 8, 10, and 12 if a jump exists only at 11. These selected jump points can be tested by using, e.g., the equality test of Chow (1960). Our allowance for a heteroskedastic error process is in part to allow us to circumvent the well-known size and power distortions of the Chow test discussed in Toyoda (1974) and Schmidt and Sickles (1977), among others.

If we have an additional observation, we can construct a shifted wavelet expansion on a second (shifted) dyadic interval. The tree-structured representation of the new coefficients can provide important information about the significance of the potential jumps detected in the first tree. Continuing

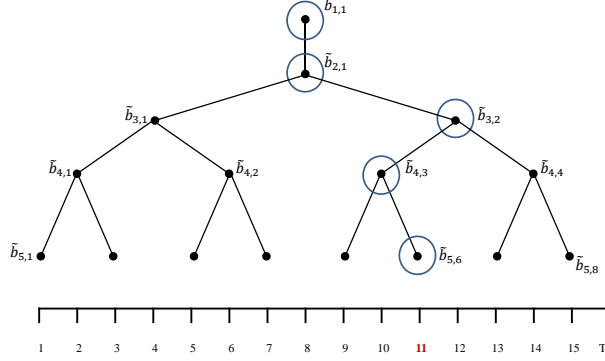


Figure 1: An illustrating example of a tree-structured representation for the wavelet coefficients.

with the same example of Figure 1, we can see that the tree-structured representations of the shifted and non-shifted coefficients presented in Figure 2 support the hypothesis of only one jump at 11.

In the multivariate case, the interpretation of the tree-structured representation can be complicated since the nodes represent vectors that contain simultaneous information about multiple regressors. In order to construct an individual tree for each parameter, we can re-transform each element of the $(\underline{P} \times 1)$ vector γ_t with the conventional univariate wavelet basis functions defined in (4). Recall that, in our differenced model, $\gamma_{t,p} = \beta_{t,p}$ and $\gamma_{t,p+P} = \beta_{t-1,p}$. This allows us to obtain for each slope parameter, β_p , two sets of univariate wavelet coefficients:

$$c_{l,k,p}^{(s)} = \frac{1}{T-1} \sum_{t=2}^T \psi_{l,k}(t-1) \gamma_{t,p}, \quad (30)$$

and

$$c_{l,k,p}^{(u)} = \frac{1}{T-1} \sum_{t=1}^{T-1} \psi_{l,k}(t) \gamma_{t+1,p+P}. \quad (31)$$

We use the superscripts (s) and (u) in (30) and (31) to denote the shifted and non-shifted coefficients, respectively.

Replacing $\gamma_{t,p}$ with $\tilde{\gamma}_{t,p} = \sum_{l=1}^L \sum_{k=1}^{K_l} \sum_{q=1}^P W_{lk,p,q}(t) \tilde{b}_{l,k,q}$ and $\gamma_{t+1,p+P}$ with $\tilde{\gamma}_{t+1,p+P} = \sum_{l=1}^L \sum_{k=1}^{K_l} \sum_{q=1}^P W_{lk,p+P,q}(t+1) \tilde{b}_{l,k,q}$, we obtain

$$\tilde{c}_{l,k,p}^{(s)} = \frac{1}{T-1} \sum_{t=2}^T \psi_{l,k}(t-1) \tilde{\gamma}_{t,p}, \quad (32)$$

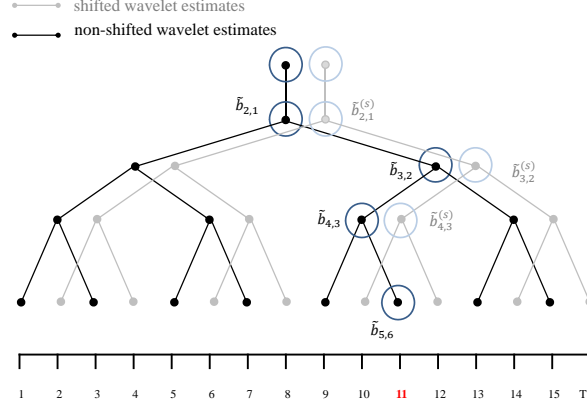


Figure 2: An illustrating example of a tree-structured representation for the shifted and non-shifted coefficients.

and

$$\tilde{c}_{l,k,p}^{(u)} = \frac{1}{T-1} \sum_{t=1}^{T-1} \psi_{l,k}(t) \tilde{\gamma}_{t+1,p+P}. \quad (33)$$

Having an appropriate threshold for $\tilde{c}_{l,k,p}^{(u)}$, we can construct the shifted and non-shifted tree-structured representation for each parameter, as before. This can provide important information about the potential spurious jumps since all low level parameters in the shifted tree fall in the highest level of the non-shifted tree and vice versa. Based on this predicate, we propose a selection method for consistently detecting the jump locations. All we need is an appropriate threshold for the highest coefficients.

The following Lemma establishes the uniform consistency in k and p of both $\tilde{c}_{L,k,p}^{(s)}$ and $\tilde{c}_{L,k,p}^{(u)}$ and states their order of magnitude in probability.

Lemma 2 *Suppose Assumptions A-C hold, then, for all $p \in \{1, \dots, P\}$ and $m \in \{m, s\}$*

$$\sup_k \left| \tilde{c}_{L,k,p}^{(m)} - c_{L,k,p}^{(m)} \right| = O_p(\sqrt{\log(T-1)/n(T-1)}).$$

From Lemma 2, we can see intuitively that asymptotically both $\tilde{c}_{L,k,p}^{(m)}$ and $\tilde{b}_{l,k,p}$ can be shrunk by the same threshold $\lambda_{n,T}$. Theorem 2 gives the necessary asymptotic conditions to ensure consistency of the jump selection method.

4.2 Detecting the Jump Locations

As mentioned earlier, interpreting all jumps of the SAW estimator as structural breaks may lead to an over-specification of the break points. In this Section, we exploit the information from the shifted and unshifted univariate wavelet coefficients (32) and (33) to construct a consistent selection method for detecting the jump locations.

We use (32) and (33) to obtain the following two estimators of $\Delta\beta_t$:

$$\Delta\tilde{\beta}_{t,p}^{(u)} = \sum_{k=1}^{K_L} \Delta\psi_{L,k}(t) \hat{c}_{L,k,p}^{(u)}, \quad \text{for } t \in \mathcal{E}, \quad (34)$$

and

$$\Delta\tilde{\beta}_{t,p}^{(s)} = \sum_{k=1}^{K_L} \Delta\psi_{L,k}(t-1) \hat{c}_{L,k,p}^{(s)}, \quad \text{for } t \in \mathcal{E}^c, \quad (35)$$

where

$$\hat{c}_{l,k,p}^{(\cdot)} = \mathbf{I}(|\tilde{c}_{l,k,p}^{(\cdot)}| > \lambda_{n,T}),$$

\mathcal{E} is the set of the even time locations $\{2, 4, \dots, T-1\}$, \mathcal{E}^c is the complement set composed of the odd time locations $\{2, 3, 4, \dots, T\} \setminus \mathcal{E}$, and $\mathbf{I}(\cdot)$ is the indicator function.

The number of jumps of each parameter can be estimated by

$$\tilde{S}_p = \sum_{t \in \mathcal{E}} \mathbf{I}(\Delta\tilde{\beta}_{t,p}^{(u)} \neq 0) + \sum_{t \in \mathcal{E}^c} \mathbf{I}(\Delta\tilde{\beta}_{t,p}^{(s)} \neq 0). \quad (36)$$

The jump locations $\tilde{\tau}_{1,p}, \dots, \tilde{\tau}_{\tilde{S}_p,p}$ can be identified as follows:

$$\tilde{\tau}_{j,p} = \min \left\{ s \left| j = \sum_{t=2}^s \mathbf{I}(\Delta\tilde{\beta}_{t,p}^{(u)} \neq 0, t \in \mathcal{E}) + \sum_{t=3}^s \mathbf{I}(\Delta\tilde{\beta}_{t,p}^{(s)} \neq 0, t \in \mathcal{E}^c) \right. \right\}, \quad (37)$$

for $j \in \{1, \dots, \tilde{S}_p\}$. The maximal number of breaks $S = \sum_{p=1}^P \tilde{S}_p$ can be estimated by $\tilde{S} = \sum_{p=1}^P \tilde{S}_p$.

Theorem 2 Under Assumptions A-C, if (c.1) : $\sqrt{\frac{n(T-1)}{\log((T-1))}} \lambda_{n,T} \rightarrow \infty$ and (c.2) : $\sqrt{T-1} \lambda_{n,T} \rightarrow 0$, as $n, T \rightarrow \infty$, then

$$(i) \lim_{n,T \rightarrow \infty} P(\tilde{S}_1 = S_1, \dots, \tilde{S}_p = S_p) = 1 \text{ and}$$

$$(ii) \lim_{n,T \rightarrow \infty} P(\tilde{\tau}_{1,1} = \tau_{1,1}, \dots, \tilde{\tau}_{\tilde{S}_p,p} = \tau_{\tilde{S}_p,p} | \tilde{S}_1 = S_1, \dots, \tilde{S}_p = S_p) = 1.$$

The crucial element for consistently estimating $\tau_{1,1}, \dots, \tau_{\tilde{S}_p,p}$ is, hence, using a threshold that converges to zero but at a rate slower than $\sqrt{\log(T-1)/(n(T-1))}$.

4.3 Post-SAW Estimation

For known $\tau_{1,p}, \dots, \tau_{S_p,p}$, we can rewrite Model (22) as

$$\Delta \dot{Y}_{it} = \sum_{p=1}^P \sum_{j=1}^{S_p+1} \Delta \dot{X}_{it,p}^{(\tau_{j,p})} \beta_{\tau_{j,p}} + \Delta \dot{e}_{it}, \quad (38)$$

where

$$\Delta \dot{X}_{(it,p)}^{(\tau_{j,p})} = \Delta \dot{X}_{it,p} \mathbf{I}(\tau_{j-1,p} < t \leq \tau_{j,p}),$$

with $\tau_{0,p} = 1$ and $\tau_{S_p+1,p} = T$, for $p \in \{1, \dots, P\}$. The dot operator transforms the variables as follows: $\dot{u}_{it} = u_{it} - \frac{1}{n} \sum_{i=1}^n u_{it}$.

Depending on the set of the jump locations $\tau := \{\tau_{j,p} | j = 1, \dots, S_p + 1, p = 1, \dots, P\}$, the vector presentation of Model (38) can be rewritten as

$$\Delta \dot{Y}_{it} = \Delta \dot{X}'_{it,(\tau)} \beta_{(\tau)} + \Delta \dot{e}_{it}, \quad (39)$$

where $\beta_{(\tau)} = (\beta_{\tau_{1,1}}, \dots, \beta_{\tau_{S_1+1,1}}, \dots, \beta_{\tau_{1,P}}, \dots, \beta_{\tau_{S_P+1,P}})'$ and $\Delta \dot{X}_{it,(\tau)} = (\Delta \dot{X}_{it,1}^{(\tau_{1,1})}, \dots, \Delta \dot{X}_{it,1}^{(\tau_{S_1+1,1})}, \dots, \Delta \dot{X}_{it,P}^{(\tau_{1,P})}, \dots, \Delta \dot{X}_{it,P}^{(\tau_{S_P+1,P})})'$.

Let $Z_{it,p}$ denote the instrument chosen for $\Delta \dot{X}_{it,p}$ and $Z_{it,(\tau)} = (Z_{it,1}^{(\tau_{1,1})}, \dots, Z_{it,1}^{(\tau_{S_1+1,1})}, \dots, Z_{it,P}^{(\tau_{1,P})}, \dots, Z_{it,P}^{(\tau_{S_P+1,P})})'$, with $Z_{it,p}^{(\tau_{j,p})} = Z_{it,p} \mathbf{I}(\tau_{j-1,p} < t \leq \tau_{j,p})$. The conventional IV estimator of $\beta_{(\tau)}$ is

$$\hat{\beta}_{(\tau)} = \left(\sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} \Delta \dot{X}'_{it,(\tau)} \right)^{-1} \left(\sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} \Delta \dot{Y}_{it} \right). \quad (40)$$

Conditional on $\tilde{S}_1 = S_1, \dots, \tilde{S}_P = S_P$, we can replace the set of the true jump locations τ in (40) with the detected jump locations $\tilde{\tau} := \{\tilde{\tau}_{j,p} | j \in \{1, \dots, S_p + 1\}, p \in \{1, \dots, P\}\}$, to obtain the post-SAW estimator:

$$\hat{\beta}_{(\tilde{\tau})} = \left(\sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tilde{\tau})} \Delta \dot{X}'_{it,(\tilde{\tau})} \right)^{-1} \left(\sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tilde{\tau})} \Delta \dot{Y}_{it} \right). \quad (41)$$

From (26) and (41), we can see that the number of parameters to be estimated after detecting the jump locations is much smaller than the number of parameters required to estimate the slope parameters in the SAW regression ($\sum_{p=1}^P (S_p + 1) < T(\underline{P} - 1)$). It is evident that such a gain in terms of regression dimension improves the quality of the estimator.

Assumption E - Central Limits: Let $\mathcal{T}_{(\tau)}$ be a $(\sum_{p=1}^P (S_p + 1) \times \sum_{p=1}^P (S_p + 1))$ diagonal matrix with the diagonal elements $T_{1,1}, \dots, T_{S_P+1,P}$, where $T_{j,p} = \tau_{j,p} - \tau_{j-1,p} + 1$.

- (i) : $(n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} \Delta \dot{X}'_{it,(\tau)} \xrightarrow{p} Q_{(\tau)}^\circ$ where $Q_{(\tau)}^\circ$ is a full rank finite matrix.
- (ii) : $(n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T \sum_{j=1}^n \sum_{s=2}^T Z_{it,(\tau)} Z'_{js,(\tau)} \sigma_{ij,ts} \xrightarrow{p} V_{(\tau)}^\circ$, where $V_{(\tau)}^\circ$ is a full rank finite matrix.
- (iii) : $(n\mathcal{T}_{(\tau)})^{-\frac{1}{2}} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} \Delta \dot{e}_{it} \xrightarrow{d} N(0, V_{(\tau)}^\circ)$.

Assumption E presents standard assumptions that are commonly used in the literature on instrumental variables.

Theorem 3 *Suppose Assumptions A-E hold. Then conditional on $\tilde{S}_1 = S_1, \dots, \tilde{S}_p = S_p$, we have*

$$\sqrt{n}\mathcal{T}_{(\tau)}^{\frac{1}{2}}(\hat{\beta}_{(\bar{\tau})} - \beta_{(\tau)}) \xrightarrow{d} N(0, \Sigma_{(\tau)}),$$

where $\Sigma_{(\tau)} = (Q_{(\tau)}^\circ)^{-1}(V_{(\tau)}^\circ)(Q_{(\tau)}^\circ)^{-1}$.

If $T \rightarrow \infty$ and all $T_{j,p}$ diverge proportionally to T , then $\hat{\beta}_{\tau_{j,p}}$ achieves the usual \sqrt{nT} -convergence rate. According to Theorem 3, our final estimator of the model parameters are first-order efficient: they have the same asymptotic distribution as the (infeasible) estimators that would be obtained if all jump locations were exactly known a priori and thus possess the "oracle property" in regard to these parameter estimates. The final estimator proposed by Qian and Su (2014) also shares this oracle property.

To summarize, our method essentially consists of two steps. A structure adapted wavelet (SAW) estimation followed by the post-SAW procedures. Our SAW method could be applied in a time series context by fixing n and letting T be large, in which case our approach may be considered a more parsimonious variant of the subinterval approach of Bai and Perron (1998, 2003). Although we do not pursue this issue in this paper we do show in Theorem 1 that our wavelet procedure provides (mean square) consistent parameter estimates even for fixed n and large T . In contrast, the method of Qian and Su (2014) essentially consists of three steps. In a first step they use the "naïve" estimator of β_t for every single time point t . This is used in the second step to construct appropriate weights for the adaptive Lasso procedure. In a third step they then propose a post Lasso instead of our wavelet procedure. An important point is that their method will only work if n is large. Their method does not consider cases in which n is small and T is large as we do.

Because the asymptotic variance $\Sigma_{(\tau)}$ of $\hat{\beta}_{(\bar{\tau})}$ in Theorem 3 is unknown, consistent estimators of $Q_{(\tau)}^\circ$ and $V_{(\tau)}^\circ$ are required to perform inferences. A natural estimator of $Q_{(\tau)}^\circ$ is

$$\hat{Q}_{(\bar{\tau})} = (n\mathcal{T}_{(\bar{\tau})})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\bar{\tau})} \Delta X'_{it,(\bar{\tau})}$$

and a consistent estimator of $\Sigma_{(\tau)}$ can be obtained by

$$\hat{\Sigma}_{(\bar{\tau}),j} = \hat{Q}_{(\bar{\tau})}^{-1} \hat{V}_{(\bar{\tau})}^{(c)} \hat{Q}_{(\bar{\tau})}^{-1},$$

where $\hat{V}_{(\bar{\tau})}^{(c)}$ a consistent estimator of $V_{(\bar{\tau})}^{(c)}$ that can be constructed depending on the structure of $\Delta \dot{e}_{it}$. For brevity, we distinguish only four cases:

1. The case of homoscedasticity without the presence of auto- and cross-section correlations:

$$\hat{V}_{(\bar{\tau})}^{(1)} = (n\mathcal{T}_{(\bar{\tau})})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\bar{\tau})} Z'_{it,(\bar{\tau})} \hat{\sigma}^2,$$

where $\hat{\sigma}^2 = \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \Delta \hat{e}_{it}^2$, with $\Delta \hat{e}_{it} = \Delta \dot{Y}_{it} - \Delta \dot{X}'_{it,(\bar{\tau})} \hat{\beta}_{(\bar{\tau})}$.

2. The case of cross-section heteroskedasticity without auto- and cross-section correlations:

$$\hat{V}_{(\bar{\tau})}^{(2)} = (n\mathcal{T}_{(\bar{\tau})})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\bar{\tau})} Z'_{it,(\bar{\tau})} \hat{\sigma}_i^2,$$

where $\hat{\sigma}_i^2 = \frac{1}{T-1} \sum_{t=2}^T \Delta \hat{e}_{it}^2$.

3. The case of time heteroskedasticity without auto- and cross-section correlations:

$$\hat{V}_{(\bar{\tau})}^{(3)} = (n\mathcal{T}_{(\bar{\tau})})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\bar{\tau})} Z'_{it,(\bar{\tau})} \hat{\sigma}_t^2,$$

where $\hat{\sigma}_t^2 = \frac{1}{n} \sum_{i=1}^n \Delta \hat{e}_{it}^2$.

4. The case of cross-section and time heteroskedasticity without auto- and cross-section correlations:

$$\hat{V}_{(\bar{\tau})}^{(4)} = (n\mathcal{T}_{(\bar{\tau})})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\bar{\tau})} Z'_{it,(\bar{\tau})} \Delta \hat{e}_{it}^2.$$

Proposition 2 *Under Assumptions A-E, we have, as $n, T \rightarrow \infty$, $\hat{\Sigma}_{(\bar{\tau})}^{(c)} = \Sigma_{(\tau)} + o_p(1)$, for $c = 1, 2, 3$, and 4.*

Remark 6 *If the errors (at the difference level) are autocorrelated, $V_{(\bar{\tau})}^{(c)}$ can be estimated by applying the standard heteroskedasticity and autocorrelation (HAC) robust limiting covariance estimator to the sequence $\{Z_{it,(\bar{\tau})} \Delta \hat{e}_{it}\}_{i,t}$ for $i \in \mathbb{N}^*$ and $t \in \mathbb{N}^* \setminus \{1\}$; see, e.g., Newey and West (1987). In the presence of additional cross-section correlations, one can use the partial sample method together with the Newey-West procedure as proposed by Bai (2009). A formal proof of consistency remains, in this case, to be explored.*

Based on the asymptotic distribution of $\hat{\beta}_{(\bar{\tau})}$ and the variance estimators presented in Proposition 2, usual test statistics such as t - and χ^2 -tests can be used for inference. In Appendix B, we propose, as supplementary materials, two test statistics that can be used directly for inference on our post-SAW detected jumps: a Chow-type test to examine the statistical significance of each estimated jump and a Hotelling-type test to examine whether a model with constant parameters is more appropriate for the data than a model with jumps.

5 SAW with Unobserved Multifactor Effects

If the endogeneity arises from a dynamic model such that one variable on the right hand side is the lag of the explained variable Y_{it} , one can follow the existing literature on dynamic panel models and choose one of the commonly used instruments such as Y_{it-2} , Y_{it-3} , and/or $Y_{it-2} - Y_{it-3}$; see, e.g., Anderson and Hsiao (1981), Arellano and Bond (1991), and Kiviet (1995).

In this section, we discuss two possible model extensions: the case in which endogeneity arises from an omitted factor structure; and the case in which endogeneity is due to the presence of simultaneous equations.

Presence of Multifactor Errors

There is a growing literature on large panel models that allows for the presence of unobserved time-varying individual effects having an approximate factor structure such that

$$e_{it} = \Lambda_i' F_t + \epsilon_{it},$$

where Λ_i is a $(d \times 1)$ vector of individual scores (or loadings) $\Lambda_{i1}, \dots, \Lambda_{id}$ and F_t a $(d \times 1)$ vector of d common factors F_{1t}, \dots, F_{dt} . Note that this extension provides a generalization of panel data models with additive effects and can be very useful in many application areas, especially when the unobserved individual effects are non-static over time; see, e.g., Pesaran (2006), Bai (2009), Ahn et al. (2013), Kneip et al. (2012), and Bada and Kneip (2014).

Leaving the factor structure in the error term and estimating the remaining parameters without explicitly considering the presence of a potential correlation between the observed regressors $X_{1,it}, \dots, X_{P,it}$ and the unobserved effects Λ_i and F_t may lead to an endogeneity problem caused by these omitted model components. The problem with the presence of the factor structure in the error term is that such a structure can not be eliminated by differencing the observed variables or using a simple within-transformation. Owing to the potential correlation between the observable regressors $X_{1,it}, \dots, X_{P,it}$ and the unobservable heterogeneity effects, we allow for the data generating process of $X_{p,it}$ to have the following rather

general form:

$$X_{p,it} = \vartheta'_{p,i} F_t + \Lambda'_i G_{p,t} + a_p \Lambda'_i F_t + \mu_{p,it}, \quad (42)$$

where $\vartheta_{p,i}$ is a $(d \times 1)$ vector of unknown individual scores, $G_{p,t}$ is a $(d \times 1)$ vector of unobservable common factors, a_p is a p -specific univariate coefficient, and μ_{it} is an individual specific term that is uncorrelated with ϵ_{it} , Λ_i , ϑ_i , F_t and G_t .

Rearranging (42), we can rewrite $X_{p,it}$ as

$$X_{p,it} = \vartheta_{p,i}^{*'} G_{p,t}^* + \mu_{p,it}, \quad (43)$$

where

$$\vartheta_{p,i}^{*'} = H(a_p \Lambda'_i + \vartheta'_{p,i}, \Lambda'_i), \quad (44)$$

and

$$G_{p,t}^* = H^{-1}(F'_t, G'_{p,t})', \quad (45)$$

for some $(2d \times 2d)$ full rank matrix H . The role of H is only to ensure orthonormality and identify uniquely (up to a sign change) the elements of the factor structure so that $\sum_{t=1}^T G_{p,t}^{*'} G_{p,t}^* / T$ is the identity matrix and $\sum_{i=1}^n \vartheta_{p,i}^{*'} \vartheta_{p,i}^* / n$ is a diagonal matrix with ordered diagonal elements.

We can see from (42) that an ideal candidate for instrumenting $X_{p,it}$ is $\mu_{p,it}$. Since $\mu_{p,it}$ is unobserved, a feasible instrument can be obtained by

$$Z_{p,it} = X_{p,it} - \hat{\vartheta}_{p,i}^{*'} \hat{G}_{p,t}^*, \quad (46)$$

where $\hat{G}_{p,t}^{*'}$ is the t -th row element of the $(2d \times 1)$ matrix containing the eigenvectors corresponding to the ordered eigenvalues of the covariance matrix of $X_{p,it}$ and $\hat{\vartheta}_{p,i}^{*'}$ is the projection of $\hat{G}_{p,t}^{*'}$ on $X_{p,it}$. If d is unknown, one can estimate the dimension of $\vartheta_{p,i}^{*'} G_{p,t}^*$ by using an appropriate panel information criterion; see, e.g., Bai and Ng (2002) and Onatski (2010). A crucial assumption about the form of dependency in $\mu_{p,it}$ is that, for all T and n , and every $i \leq n$ and $t \leq T$,

1. $\sum_{s=1}^T |E(\mu_{p,it} \mu_{p,is})| \leq M$ and
2. $\sum_{k=1}^n |E(\mu_{p,it} \mu_{p,kt})| \leq M$.

Bai (2003) proves the consistency of the principal component estimator when additionally $\frac{1}{T} \sum_{t=1}^T G_{p,t}^{*'} G_{p,t}^* \xrightarrow{p} \Sigma_{G_p^*}$ for some $(2d \times 2d)$ positive definite matrix $\Sigma_{G_p^*}$, $\|\vartheta_{p,i}^*\| \leq M$ for all i and p , and $\|\frac{1}{n} \sum_{i=1}^n \vartheta_{p,i}^{*'} \vartheta_{p,i}^* - \Sigma_{\vartheta_p^*}\| \rightarrow 0$, as $n \rightarrow \infty$ for some $(2d \times 2d)$ positive definite matrix $\Sigma_{\vartheta_p^*}$.

By instrumenting $X_{p,it}$ with $Z_{p,it}$ in (46), we can consistently estimate the jumping slope parameters as before.

Two-Step SAW for Jump Reverse Causality

Besides the issues of omitted variables and dynamic dependent variables, another important source of endogeneity is the phenomenon of reverse causality. This occurs when the data, e.g., is generated by a system of simultaneous equations.

Consider the following two-equation simultaneous equation system:

$$Y_{it} = \mu + \sum_{p=1}^P X_{p,it} \beta_{t,p} + \alpha_i + \theta_t + e_{it}, \quad (47)$$

and

$$X_{q,it} = b_t Y_{it} + \sum_{p \in \{1, \dots, P\} \setminus \{q\}} X_{p,it} d_{t,p} + v + u_i + \vartheta_t + \nu_{it}, \quad (48)$$

for some a $q \in \{1, \dots, P\}$, where $b_t \neq 1/\beta_{t,q}$, and the parameters v, u_i , and ϑ_t are unknown parameters.

Neglecting the structural form of $X_{q,it}$ in Equation (48) and estimating the regression function (47) without instrumenting this variable results in an inconsistent estimation since $X_{q,it}$ and e_{it} are correlated (due to the presence of Y_{it} in Equation (48)). A natural way to overcome this type of endogeneity problem is to use the fitted variable obtained from Equation (48) as an instrument after replacing Y_{it} with its expression in (47). However, our model involves an additional complication related to the time-changing character of $\beta_{t,q}$ and the presence of the unobservable heterogeneity effects that render such two-stage least squares estimators problematic. Inserting (47) in (48) and rearranging it leads to a panel model with time-varying unobservable individual effects:

$$X_{q,it} = \sum_{p \in \{1, \dots, P\} \setminus \{q\}} X_{p,it} d_{t,p}^* + \vartheta_{1t}^* + u_i \vartheta_{2t}^* + \alpha_i \vartheta_{3t}^* + \varepsilon_{it}, \quad (49)$$

where

$$\begin{aligned} d_{t,p}^* &= b_t \beta_{t,p} + d_{t,p}, \\ \vartheta_{1t}^* &= \frac{b_t \mu + b_t \theta_t + \vartheta_t + v}{1 - b_t \beta_{t,p}}, \\ \vartheta_{2t}^* &= \frac{1}{1 - b_t \beta_{t,p}}, \\ \vartheta_{3t}^* &= \frac{b_t \mu + b_t \theta_t + \vartheta_t + v}{1 - b_t \beta_{t,p}}, \text{ and} \\ \varepsilon_{it} &= b_t e_{it} + \varepsilon_{it}. \end{aligned}$$

Note that the regression model in (49) can be considered a special case of the model with multifactor errors discussed above. A potential instrument for $X_{q,it}$ in (47) is then

$$Z_{q,it} = \sum_{p \in \{1, \dots, P\} \setminus \{q\}} X_{p,it} \hat{d}_{t,p}^* + \hat{\vartheta}_i' \hat{G}_t, \quad (50)$$

where $\hat{d}_{t,p}^*$ and $\hat{\vartheta}_i' \hat{G}_t$ are the estimators of b_t and $\vartheta_i' G_t = \vartheta_{1t}^* + u_i \vartheta_{2t}^* + \alpha_i \vartheta_{3t}^*$, respectively, and which can be obtained from (49) by using the instruments proposed above to control for the omitted factor structure $\vartheta_{1t}^* + u_i \vartheta_{2t}^* + \alpha_i \vartheta_{3t}^*$.

6 Monte Carlo Simulations

In this section we examine, through Monte Carlo simulations, the finite sample performance of our method. Our data generating-processes are based on the following panel data model:

$$Y_{it} = X_{it}\beta_t + \alpha_i + \sqrt{\theta_{it}}e_{it} \text{ for } i \in \{1, \dots, n\} \text{ and } t \in \{1, \dots, T\},$$

where

$$\beta_t = \begin{cases} \beta_{\tau_1} & \text{for } t \in \{1, \dots, \tau_1\}, \\ \vdots & \\ \beta_{\tau_S+1} & \text{for } t \in \{\tau_S + 1, \dots, T\}, \end{cases} \quad (51)$$

with

$$\beta_{\tau_j} = \frac{2}{3} \cdot (-1)^j \text{ and } \tau_j = \left\lfloor \frac{j}{S+1}(T-1) \right\rfloor, \text{ for } j = 1, \dots, S+1.$$

We examine the situations where the number of jumps is $S = 0, 1, 2, 3$. In the no-jump case ($S = 0$), we compare the performance of our method with the performance of the classical Least Squares Dummy Variable Method (LSDV), the Generalized Least Squares Method for random effect models (GLS), the Iterated Least Squares Method (ILS) of Bai (2009), and the semi-parametric method (KSS) of Kneip et al. (2012). Our thresholding parameter is calculated with $\kappa = 1 - \log(\log(nT))/\log(nT)$. To see how the properties of the estimators vary with n and T , we consider 12 different combinations with the sizes $n = 30, 60, 120, 300$ and $T = 2^{L-1} + 1$, for $L = 6, 7, 8$, i.e., $T = 33, 65, 129$. We consider the cases of dyadic (e.g., when $S = 1$ and $\tau_1 = (T-1)/2$) and non-dyadic jump locations (when $S = 2, 3$) as well as models with exogenous and endogenous regressors.

The rationale for these values is simply related to the DGP's below that we use for generating the jump locations. According to our DGP if $S = 1$ then τ_1 is dyadic $(T-1)/2$ for $T = 33, 65$, and, 129. When $S = 2, 3$ then the generated τ_j are by construction at non-dyadic locations.

Our experiments are based on the results of seven different DGP-configurations:

DGP1 (*exogeneity, and i.i.d. errors*): the dependent variable X_{it} is uncorrelated with e_{it} and generated by

$$X_{it} = 0.5\alpha_i + \xi_{it}, \quad (52)$$

with $\xi_{it}, \alpha_i, e_{it} \sim N(0, 1)$ and $\theta_{it} = 1$ for all i and t .

DGP2 (*exogeneity, and cross-section heteroskedasticity*): the DGP of the exogenous regressor X_{it} is of form (52); cross-sectionally heteroskedastic errors such that $e_{it} \sim N(0, 1)$ with $\theta_{it} = \theta_i^* \sim U(1, 4)$ for all t

DGP3 (*exogeneity, and heteroskedasticity in time and cross-section dimension*): the DGP of the exogenous regressor X_{it} is of form (52); heteroskedastic errors in time and cross-section dimension such that $e_{it} \sim N(0, 1)$ and $\theta_{it} \sim U(1, 4)$.

DGP4 (*exogeneity, and serial correlation with cross-section heteroskedasticity*): the DGP of the exogenous regressor X_{it} is of form (52); homoscedasticity and autocorrelation in the errors such that

$$e_{it} = \rho_i e_{it-1} + \zeta_{it}, \quad (53)$$

with $\rho_i \sim U(0, .5)$, $\zeta_{it} \sim N(0, .5)$, and $\theta_{it} = 1$ for all i and t .

DGP5 (*endogeneity due to a hidden factor structure*): X_{it} and e_{it} are correlated through the presence of a hidden factor structure:

$$\begin{aligned} e_{it} &= \lambda_i f_t + \epsilon_{it} \text{ and} \\ X_{it} &= 0.3\alpha_i + 0.3\nu_t + 0.3\lambda_i f_t + \mu_{it}, \end{aligned} \quad (54)$$

with $\lambda_i, f_t \sim N(0, .5)$, $\theta_{it} = 1$ for all i and t , and $\alpha_i \sim N(0, 1)$.

DGP6 (*endogeneity due to a hidden approximate factor structure*): X_{it} and e_{it} are correlated as in DGP5, but

$$\begin{aligned} \epsilon_{it} &= \rho_{e,i} \epsilon_{i,t-1} + \zeta_{e,it}, \\ \mu_{it} &= \rho_{\mu,i} \mu_{i,t-1} + \zeta_{\mu,it}, \end{aligned} \quad (55)$$

with $\zeta_{e,it}, \zeta_{\mu,it} \sim N(0, .5)$, $\rho_{e,i}, \rho_{\mu,i} \sim U(0, .5)$, $\theta_{it} = 1$ for all i and t , and $\alpha_i \sim N(0, 1)$.

DGP7 (*no-jumps, endogeneity, and hidden approximate factor structure*): the slope parameter does not suffer from structural breaks so that $\beta_t = 2$ for all t ; the regressor and the error are correlated through the presence of an approximate factor structure as in DGP6.

Tables 1-4 report the estimation results obtained by averaging the results of 1000 replications. The third, sixth, and ninth columns in Tables 1-3 report the averages of the estimated number of jumps \tilde{S} detected by (36) for $S = 1, 2$, and 3 , respectively. The MISE of our estimator is calculated by $\frac{1}{1000} \sum_{r=1}^{1000} (\frac{1}{T} \sum_{t=1}^T (\hat{\beta}_t^r - \beta_t)^2)$, where $\hat{\beta}_t^r$ is the pointwise post-SAW estimate of β_t obtained in replication r . The fourth, seventh, and tenth columns in Tables 1-3 give, on average, the values of a criterion (hereafter called MDCJ)

that describes the mean distance between the true jump locations and the closest post-SAW detected jumps. The MDCJ criterion is calculated as follows:

$$\text{MDCJ} = \frac{1}{S} \sum_{j=1}^S \min_{l \in \{1, \dots, \tilde{S}\}} |\tau_j - \tilde{\tau}_l|.$$

We use the R-package `phtt` to calculate LSDV, ILS, and KSS and `plm` to calculate GLS. The corresponding MSEs of LSDV, GLS, ILS, and KSS are obtained by $\frac{1}{1000} \sum_{r=1}^{1000} (\hat{\beta}_{(M)}^r - \beta)^2$, where $\hat{\beta}_{(M)}^r$ is the estimate of $\beta = \beta_1 = \dots = \beta_T$ obtained in replication r by using method $M = \text{LSDV, ILS, and KSS}$. The results are reported in Table 4.

DGP1										
Nbr. of jumps S :			1		2			3		
n	T	\tilde{S}	MDCJ	MISE	\tilde{S}	MDCJ	MISE	\tilde{S}	MDCJ	MISE
30	33	1.0	0.000	0.002	2.0	0.000	0.005	3.0	0.000	0.008
60	33	1.0	0.000	0.001	2.0	0.000	0.004	3.0	0.000	0.008
120	33	1.0	0.000	0.001	2.0	0.000	0.004	3.0	0.000	0.007
300	33	1.0	0.000	0.001	2.0	0.000	0.003	3.0	0.000	0.007
30	65	1.0	0.000	0.001	2.1	0.000	0.001	3.1	0.000	0.002
60	65	1.0	0.000	0.000	2.0	0.000	0.001	3.0	0.000	0.002
120	65	1.0	0.000	0.000	2.0	0.000	0.001	3.0	0.000	0.002
300	65	1.0	0.000	0.000	2.0	0.000	0.001	3.0	0.000	0.002
30	129	1.0	0.000	0.000	2.1	0.000	0.000	3.1	0.000	0.002
60	129	1.0	0.000	0.000	2.0	0.000	0.000	3.0	0.000	0.001
120	129	1.0	0.000	0.000	2.0	0.000	0.000	3.0	0.000	0.000
300	129	1.0	0.000	0.000	2.0	0.000	0.000	3.0	0.000	0.000

DGP2										
30	33	0.9	3.100	0.118	1.5	2.731	0.181	2.2	2.349	0.193
60	33	1.0	0.000	0.003	2.0	0.111	0.011	3.0	0.053	0.013
120	33	1.0	0.000	0.002	2.0	0.000	0.005	3.0	0.000	0.008
300	33	1.0	0.000	0.001	2.0	0.000	0.003	3.0	0.000	0.008
30	65	0.8	9.200	0.173	1.5	5.470	0.191	2.4	4.160	0.180
60	65	1.0	0.000	0.001	1.8	0.665	0.021	2.9	0.531	0.030
120	65	1.0	0.000	0.001	2.0	0.000	0.001	3.0	0.000	0.002
300	65	1.0	0.000	0.000	2.0	0.000	0.001	3.0	0.000	0.002
30	129	0.9	13.81	0.124	1.4	16.40	0.261	2.0	12.31	0.231
60	129	1.0	2.519	0.021	2.0	0.859	0.017	2.9	0.851	0.022
120	129	1.0	0.000	0.000	2.0	0.000	0.000	3.0	0.000	0.001
300	129	1.0	0.000	0.000	2.0	0.000	0.000	3.0	0.000	0.001

Table 1: Simulation results of the Monte Carlo experiments for DGP1-DGP2. The entries are the averages of 1000 replications.

In our examined data configurations, the MISE of the post-SAW estimator and the average of the estimated number of jumps behave properly as both n and T become large as well as when T is fixed and only n becomes large. The method performs quite well in the benchmark case where idiosyncratic errors are independent and identically distributed even when n and T are relatively small (e.g., the combinations where $n = 30$ and/or $T = 33$ in the first part of Table 1). In most of the examined cases, where heteroskedasticity in the cross-section and time dimension and/or week serial correlations exist, the method still behaves very well, in particular when n is large (see results of DGP3-DGP4 in Tables 1 and 2). The quality of the estimator seems to be independent of the number and the location of the jumps (i.e., dyadic, for $S = 1$, and non-dyadic for $S = 2, 3$). Not surprisingly, the jump selection method performs poorly when n is fixed and only

DGP3										
Nbr. of jumps S :		1			2			3		
n	T	\bar{S}	MDCJ	MISE	\bar{S}	MDCJ	MISE	\bar{S}	MDCJ	MISE
30	33	0.9	3.000	0.117	1.5	2.730	0.180	2.3	2.347	0.190
60	33	1.0	0.000	0.003	2.0	0.110	0.014	3.0	0.053	0.016
120	33	1.0	0.000	0.002	2.0	0.000	0.005	3.0	0.000	0.008
300	33	1.0	0.000	0.001	2.0	0.000	0.003	3.0	0.000	0.008
30	65	0.7	9.300	0.170	1.5	5.470	0.191	2.4	4.160	0.181
60	65	1.0	0.000	0.001	1.9	0.660	0.025	2.9	0.533	0.031
120	65	1.0	0.000	0.001	2.0	0.000	0.001	3.0	0.000	0.002
300	65	1.0	0.000	0.000	2.0	0.000	0.001	3.0	0.000	0.002
30	129	0.9	13.80	0.124	1.3	16.41	0.260	2.0	12.37	0.235
60	129	1.0	2.520	0.023	2.0	0.860	0.016	2.9	0.853	0.023
120	129	1.0	0.000	0.000	2.0	0.000	0.000	3.0	0.000	0.001
300	129	1.0	0.000	0.000	2.0	0.000	0.000	3.0	0.000	0.001

DGP4										
30	33	0.2	12.60	0.477	0.5	8.250	0.431	0.6	8.673	0.499
60	33	0.4	9.000	0.340	1.0	5.280	0.309	1.4	5.280	0.360
120	33	0.9	1.500	0.059	1.8	1.210	0.087	2.6	1.013	0.116
300	33	1.0	0.000	0.002	2.0	0.000	0.004	3.0	0.000	0.009
30	65	0.3	23.22	0.426	0.4	17.49	0.453	0.5	17.72	0.488
60	65	0.6	13.02	0.238	0.9	11.49	0.318	1.2	11.62	0.403
120	65	0.9	4.340	0.080	1.8	2.190	0.073	2.6	2.560	0.110
300	65	1.0	0.000	0.001	2.0	0.000	0.001	3.0	0.000	0.003
30	129	0.1	55.44	0.496	0.3	37.41	0.472	0.4	36.05	0.516
60	129	0.5	34.02	0.305	0.8	26.66	0.377	1.0	26.88	0.427
120	129	0.8	10.08	0.091	1.7	6.880	0.116	2.5	6.187	0.146
300	129	1.0	0.000	0.000	2.0	0.000	0.001	3.0	0.000	0.001

Table 2: Simulation results of the Monte Carlo experiments for DGP3-DGP4. The entries are the averages of 1000 replications.

T is large. In such a case, the threshold under-estimates the true number of jumps and the MDCJ increases with T . This effect vanishes properly as n gets large.

Table 3 reports the results of our experiments when the regressors are affected by an omitted factor structure in the error term. The proposed two-step SAW procedure seems to perform very well even when heteroskedasticity in the cross-section and time dimension and/or week serial correlations are present.

The goal of examining DGP7 is to test whether SAW is also able to detect the no-jump case. The results from Table 4 speak to this question. In the no-jump case our method is slightly inferior in terms of MSE than ILS but better than LSDV, GLS, and KSS. Because LSDV and GLS neglect the presence of the factor structure in the model and KSS is only appropriate for factors that possess smooth patterns over time, the MSEs of these three estimators are affected by a small bias that seems to persist even when n and T get large.

The Monte Carlo experiments show that, in many configurations of the data, our method performs very well even when the idiosyncratic errors are weakly affected by serial-autocorrelation and/or heteroskedasticity, independently of the number and locations of the jumps.

DGP5										
Nbr. of jumps S :		1			2			3		
n	T	\bar{S}	MDCJ	MISE	\bar{S}	MDCJ	MISE	\bar{S}	MDCJ	MISE
30	33	0.8	4.100	0.187	1.3	3.230	0.250	2.1	3.367	0.291
60	33	1.0	0.000	0.004	2.0	0.119	0.017	3.0	0.058	0.020
120	33	1.0	0.000	0.003	2.0	0.000	0.007	3.0	0.000	0.010
300	33	1.0	0.000	0.002	2.0	0.000	0.002	3.0	0.000	0.009
30	65	0.7	9.700	0.210	1.4	5.976	0.210	2.4	4.860	0.211
60	65	1.0	0.000	0.002	1.9	0.690	0.031	2.9	0.539	0.038
120	65	1.0	0.000	0.001	2.0	0.000	0.001	3.0	0.000	0.002
300	65	1.0	0.000	0.000	2.0	0.000	0.001	3.0	0.000	0.002
30	129	0.8	19.80	0.224	1.2	21.41	0.361	2.0	19.37	0.315
60	129	1.0	2.611	0.053	2.0	0.952	0.017	2.8	1.153	0.033
120	129	1.0	0.000	0.000	2.0	0.000	0.000	3.0	0.000	0.001
300	129	1.0	0.000	0.000	2.0	0.000	0.000	3.0	0.000	0.001

DGP6										
30	33	0.2	11.66	0.448	0.5	8.250	0.427	0.6	8.320	0.498
60	33	0.5	7.200	0.272	1.0	5.390	0.306	1.5	4.907	0.370
120	33	0.9	0.900	0.036	1.8	0.990	0.074	2.6	1.013	0.114
300	33	1.0	0.000	0.002	2.0	0.000	0.005	3.0	0.000	0.009
30	65	0.3	22.94	0.419	0.3	18.77	0.484	0.6	16.93	0.480
60	65	0.5	14.88	0.272	1.1	10.44	0.304	1.1	13.76	0.401
120	65	0.8	5.580	0.102	1.8	1.980	0.070	2.6	2.667	0.116
300	65	1.0	0.000	0.001	2.0	0.000	0.001	3.0	0.000	0.003
30	129	0.1	54.74	0.493	0.1	40.53	0.487	0.3	38.61	0.523
60	129	0.3	42.84	0.383	0.7	27.52	0.372	1.0	26.45	0.432
120	129	0.8	11.34	0.102	1.8	5.160	0.091	2.4	7.680	0.157
300	129	1.0	0.000	0.000	2.0	0.000	0.000	3.0	0.000	0.001

Table 3: Simulation results of experiments for DGP5-DGP6. The entries are the averages of 1000 replications.

7 Application: Algorithmic Trading and Market Quality

An issue of increasing debate, both academically and politically, is the impact of algorithmic trading (AT) on standard measures of market quality such as liquidity and volatility. Proponents, including many of the exchanges themselves, argue that AT provides added liquidity to markets and is beneficial to investors. Opponents instead caution that AT increases an investor’s perception that an AT partner possesses an informational advantage and thus may undermine investors’ beliefs that markets are in fact “fair”. Additionally, there are concerns that the benefits from AT are transient. That is, AT provides “phantom” liquidity that may disappear at a moment’s notice. Incidents such as the “flash crash” of May, 2010, although anecdotal, do nothing to alleviate these fears.

From a regulatory standpoint AT has received wide attention, both in US markets and abroad. A number of initiatives have been put forth by regulatory agencies. In 2012 the US Securities and Exchange Commission (SEC) put forth rule 613 (“The Consolidated Audit Trail”), which requires exchanges to essentially track the footprint of every order put into the system. Similarly, the “Large Trader Reporting Rule”, put forth in 2011, imposes certain reporting requirements on large traders in order for the SEC to monitor their trading patterns. Similar legislation has been proposed in the European markets in the form of the Markets in Financial Instruments

DGP7 ($S = 0$)							
Method		post-SAW	LSDV	GLS	ILS	KSS	
n	T	\bar{S}	MISE	MSE	MSE	MSE	MSE
30	33	0.0000	0.0062	0.0105	0.0148	0.0007	0.0104
60	33	0.0000	0.0041	0.0105	0.0142	0.0006	0.0101
120	33	0.0000	0.0012	0.0090	0.0125	0.0002	0.0085
300	33	0.0000	0.0004	0.0093	0.0128	0.0001	0.0090
30	65	0.0000	0.0039	0.0105	0.0135	0.0004	0.0099
60	65	0.0000	0.0015	0.0102	0.0130	0.0002	0.0101
120	65	0.0000	0.0005	0.0099	0.0126	0.0001	0.0098
300	65	0.0000	0.0001	0.0108	0.0137	0.0000	0.0108
30	129	0.0000	0.0015	0.0101	0.0129	0.0002	0.0102
60	129	0.0000	0.0008	0.0103	0.0127	0.0001	0.0104
120	129	0.0000	0.0002	0.0101	0.0125	0.0000	0.0101
300	129	0.0000	0.0000	0.0090	0.0112	0.0000	0.0090

Table 4: Simulation results of the Monte Carlo experiments for DGP7. The entries are the averages of 1000 replications.

Directive (MiFiD II). An extensive discussion is presented in Shorter and Miller (2014).

Given the scrutiny placed on AT, along with the uncertain effects, thorough empirical analysis is required. Recent work examining the effects of AT on market quality have generally found its presence to be beneficial in the sense that standard measures of liquidity such as bid-ask spreads and price-impact are negatively correlated with measures of AT. For example Hendershott et al. (2011) find that, with the exception of the smallest quintile of NYSE stocks, increases in AT are almost universally correlated with decreases in quoted and effective spreads in the remaining quintiles. Hasbrouck and Saar (2013) find similarly compelling evidence using a measure of AT constructed from order level data. A drawback of both approaches, and more specifically of the standard panel regression approach, is that estimates of the marginal effects of AT on spreads are necessarily averaged over all possible states of the market. This is problematic from an asset pricing perspective.

Of particular importance to the concept of liquidity is the timing of its provision. The merits of added liquidity during stable market periods at the expense of its draw back during periods of higher uncertainty are ambiguous without a valid welfare analysis and can potentially leave investors worse off. The issue of timing is particularly important for empirical work examining the effects of AT on market quality. Samples are often constrained in size due to limitations on the availability of data and computational concerns. As noted by Hendershott et al. (2011), it may be because samples often used do not cover large enough periods of market turbulence that detection of possible negative effects of AT on market quality have not been empiri-

cally documented. Additionally, standard sub-sample analysis requires the econometrician to diagnose market conditions as well their start and end dates, in effect imposing their own prior beliefs on the factors that might cause variation in the marginal effects.

In order to address these many issues that hinge on the stability of the relationship between algorithmic trading and market quality, we propose the use of a new estimator to automatically detect jumps in such a relationship. Specifically, in the context of a two-way panel model, we provide estimates of the marginal effects of AT on various liquidity measures while allowing those effects to change at unknown times over the sample period. The estimation procedure automatically detects the unknown breakpoints and provides point estimates of marginal effects at all breakpoints. This methodology alleviates concerns about ad-hoc sub-sample selection. It also provides key insight into the dynamics between AT and liquidity through the analysis of periods where the effects vary. Furthermore, a more careful evaluation of the breakpoints may provide valuable insight for future studies (both theoretical and empirical) and policy recommendations regarding the regulation of trading in financial markets.

7.1 Liquidity and Asset Pricing

Before discussing the effects of liquidity on asset pricing, we first examine conventional tests that assume constant parameters. In this simple example, we consider a regression of a measure of market quality on an AT proxy for an individual stock using the following model:

$$MQ_t = \alpha + AT_t \bar{\beta} + e_t, \quad (56)$$

where the time index $t \in \{1, \dots, T\}$. If the slope parameter is time varying then $\bar{\beta}$ in (56) presents only the time average of the true parameter, say β_t . In this case the conventional estimator of $\bar{\beta}$ is consistent only under the assumption $\sum_{t=1}^T AT_t^2 (\beta_t - \bar{\beta}) / T \xrightarrow{p} 0$, as T gets large. Even when such a requirement is satisfied, the *average* effect is not the correct measure to consider when the question is whether AT is beneficial to market quality, as we explain below.

An asset's expected return (i.e., the risk premium) is a function of its covariance with the stochastic discount factor (SDF). While the form of the SDF depends on the asset pricing model one is considering, it can in general be thought of as the ratio of the marginal value of wealth between time $t+1$ and t . Therefore, holding other things constant, if an asset pays off more in states in which the marginal value of wealth is relatively low and less in states where the marginal value of wealth is relatively high then a rational investor would discount the price of that asset more heavily. Thus, if an asset's return contains a stochastic liquidity component then its covariance with the SDF can have a substantial impact on pricing.

The model of Acharya and Pedersen (2005) is particularly relevant as it exemplifies the many avenues through which time varying liquidity can affect expected returns. Using an overlapping generations model they decompose conditional security returns into five components: one related to the expected level of illiquidity and four others related to terms involving the covariances between the market return, market illiquidity, asset returns and asset illiquidity. They show that asset returns are increasing in the covariance between portfolio illiquidity and market illiquidity and decreasing in the covariance between asset illiquidity and the market return. A consequence of this is that if AT intensifies these liquidity dynamics for a particular asset then the effect will be to increase the risk premium associated with that security. Increased risk premiums represent higher costs of capital for firms and thus increased AT can potentially decrease firm investment (relative to a market with no AT) through its effect on liquidity dynamics.

7.2 Data

Our sample consists of a balanced panel of stocks whose primary exchange is the New York Stock Exchange (NYSE) and covers the calendar period 2003–2008. To build measures of market quality, we use the NYSE Trade and Quotation Database (TAQ) provided by Wharton Research Data Services (WRDS) to collect intra-day data on securities. We construct various daily liquidity measures and then average those over the course of the month to construct our sample. This allows us to compare our findings with results that have appeared in the literature and that are also based on monthly data. The full sample consists of 378 firms over 71 months for each firm.² We merge the TAQ data with information on price and shares outstanding from the Center for Research in Security Prices (CRSP). The choice of this sample period reflects our desire to include both relatively stable and turbulent market regimes. We are limited in our choice of sample periods by the fact that *AT* is a recent phenomenon and that our estimation procedure requires a balanced panel. We additionally filter out a number of firms due to infrequent intra-day data.

7.2.1 The Algorithmic Trading Proxy

Our AT proxy is motivated by Hendershott et al. (2011) and Boehmer et al. (2012), who note that AT is generally associated with an increase in order

²One month is lost due to the use of lagged market quality measures in the regression specification. Furthermore, due to the sample size being very close to a dyadic integer we utilize only the final 64 months when applying the SAW estimator to our data set. As is clear in the analysis, all evidence of structural breaks occurs toward the end of the sample and thus we view this as a reasonable approach.

activity at smaller dollar volumes. Thus the proxy we consider is the negative of dollar volume (in hundreds of dollars, $\$Vol_{it}$) over time period t divided by total order activity over time period t . We define order activity as the sum of trades (Tr_{it}) and updates to the best prevailing bid and offer (q_{it}) on the security's primary exchange:

$$AT_{it} = -\frac{\$Vol_{it}}{Tr_{it} + q_{it}}.$$

An increase in AT_{it} represents a decrease in the average dollar volume per instance of order activity and represents an increase in the AT in the particular security. For example, an increase of 1 unit of AT_{it} represents a decrease of \$100 of trading volume associated with each instance of order activity (trade or quote update).

Our proxy, like that in Boehmer et al. (2012), differs from the proxy in Hendershott et al. (2011) since the latter have access to the full order book of market makers whereas we only have access to the trades and the best prevailing bid and offers of market makers through TAQ. We appeal to the same argument as Boehmer et al. (2012) in that many AT strategies are generally executed at the best bid and offer rather than behind it. Therefore, we feel our proxy is in general representative of the full order book.

7.2.2 Market Quality Measures

We consider several common measures of market quality to assess the impact of AT on markets for individual securities.

Proportional Quoted Spread

The proportional quoted spread (PQS_{it}) measures the quoted cost as a percentage of the price (Bid-Offer midpoint) of executing a trade in security i and is defined as,

$$PQS_{it} = 100 \left(\frac{Ofr_{it} - Bid_{it}}{0.5(Ofr_{it} + Bid_{it})} \right).$$

We multiply by 100 in order to place this metric in terms of percentage points. We aggregate this metric to a monthly quantity by computing a share volume-weighted average over the course of each month. An increase in PQS_{it} represents a decrease in the amount of liquidity in the market for security i due to increased execution costs.

Proportional Effective Spread

The proportional effective spread (PES_{it}) is quite similar to (PQS_{it}) but accommodates potentially hidden liquidity or stale quotes by evaluating the

actual execution costs of a trade. It is defined as,

$$\text{PES}_{it} = 100 \left(\frac{|P_{it} - M_{it}|}{M_{it}} \right),$$

where P_{it} is the price paid for security i at time t and M_{it} is the midpoint of the prevailing bid and ask quotes for security i at time t . Thus, PES_{it} is the actual execution cost associated with every trade. We again aggregate this measure up to a monthly quantity in the same way as we do for quoted spreads. Like PQS_{it} , PES_{it} is also in terms of percentage points. An increase in PES_{it} represents a decrease in the amount of liquidity in the market for security i due to increased execution costs.

Measures of Volatility

We also consider two different measures of price volatility in security i over time period t . The first is the daily high-low price range given by,

$$\text{H-L}_{it} = 100 \left(\frac{\max_{\tau \in t}(P_{it}) - \min_{\tau \in t}(P_{it})}{P_{it}} \right),$$

which represents the extreme price disparity over the course of a trading day. We also consider the realized variance of returns over each day computed using log percentage (i.e. $\ln(p_{i,t}/p_{i,t-1}) \times 100$) returns over 5-minute intervals:

$$RV_{it} = \left(\sum_{\tau \in t} r_{i\tau}^2 \right).$$

Realized variance is a nonparametric estimator of the integrated variance over the course of a trading day (see, for example, Andersen et al. (2003)). We aggregate both measures up to a monthly level by averaging over the entire month. We additionally represent each in terms of percentages, i.e. $H - L_{it}$ is the price range as a percentage of the daily closing price and RV_{it} is an estimate of the integrated variance of log returns in percentages. Both measures represent a measure of the price dispersion over the course of the trading month.

7.2.3 Additional Control Variables

While we attempt to determine the effect that our AT proxy has on measures of market quality we include in all our regressions a vector of control variables to isolate the effects of AT independent of the state of the market. We lag the control variables by one month so that they represent the state of the market at the beginning of the trading month in question. The control variables are: (1) Share Turnover (ST_{it}), which is the number of shares traded over the course of a day in a particular stock relative to the total amount of

shares outstanding; (2) Inverse price, which represents transaction costs due to the fact that the minimum tick size is 1 cent; (3) Log of the market value of equity to accommodate effects associated with smaller securities; (4) Daily price range to accommodate any effects from large price swings in the previous month. To avoid adding lagged dependent variables in the model, for regressions where the daily price range is the dependent variable we replace it in the vector of controls with the previous month's realized variance. We additionally include security and time period fixed effects to proxy for any time period or security related effects not captured by our included variables.

Potential Endogeneity Issue

We assume a linear relationship between our measures of market quality, our proxy for algorithmic trading and our control variables,

$$MQ_{it} = \mu + \alpha_i + \gamma_t + AT_{it}\beta_t + X'_{it}\delta + \epsilon_{it}. \quad (57)$$

The key distinction between the model considered here and others in the literature is that we allow the marginal effect that AT has on market quality to be time varying. Absent a theoretical model of AT, an issue on which the literature is still somewhat agnostic, it is uncertain whether AT strategies attempt to time shocks to market quality. This creates a potential problem of endogeneity with our AT proxy. That is, when estimating the regression equation (57) our estimates may be biased ($E(AT_{it}\epsilon_{it}) \neq 0$) and inconsistent.

To overcome this potential issue we use the approach of Hasbrouck and Saar (2013) (albeit with different variables) and choose as an instrument the average value of algorithmic trading over all other firms not in the same industry as firm i . To this end, we define industry groups using 4-digit SIC codes and define these new variables as $AT_{-IND,it}$. The use of this instrument requires some commonality in the level of AT across all stocks that is sufficient to pick up exogenous variation. It further rules out trading strategies by ATs across firms in different industry groups. Lacking much knowledge of the algorithms used by AT firms we view this assumption as reasonable. As noted by Hasbrouck and Saar (2013), it is unlikely that AT firms implement cross-stock trading strategies for a particular firm with the entire universe of firms (or in our case the other 377). To the extent that AT firms do implement these cross-stock trading strategies across industries, their effect on the average is likely to be marginal.

To estimate the model we use a two-stage approach and first fit the regression model,

$$AT_{it} = a_i + g_t + bAT_{-IND,it} + dW_{it} + \epsilon_{it} \quad (58)$$

to obtain an instrument, Z_{it} , for AT_{it} given by the fitted values from (58), i.e., $Z_{it} := \hat{AT}_{it} = \hat{a}_i + \hat{g}_t + \hat{b}AT_{-IND,it} + \hat{d}W_{it}$, where \hat{a}_i , \hat{g}_t , \hat{b} , and \hat{d} are the conventional estimates of a_i , g_t , b , and d . We then carry out the second stage regression using equation (57) using the estimator discussed in Section 3.3. For comparison purposes, we additionally apply the conventional panel data model assuming a constant slope parameter, i.e., $\beta_1 = \beta_2 = \dots = \beta_T$.

7.3 Results

Table 5 presents the results from a baseline model that assumes the slope parameters are constant over time.³ These results are largely consistent with previous studies that find a positive (in terms of welfare) average relationship between AT and measures of market quality over the time period considered. The coefficient estimates on the AT proxy are negative and significant for all four measures of market quality that we consider. That is, increases in AT generally reduce both of the spread measures and both of the variance measures we consider. As for as the direction of the effect, differences in our proxies and choice of instruments do not seem to reach conclusions that are at variance with the prior literature.

To gauge the size of this effect we note that the within-standard deviation of our AT proxy, after being scaled by 100, is 0.18. Combining this with the coefficient estimates from Table 5, this implies that a one standard deviation increase in AT results in a reduction of quoted spreads (effective spreads) of approximately 0.002% (0.001%). On an absolute level these effects are small. For example, given a hypothetical stock with an initial price of \$100, a one standard deviation increase in AT would reduce the quoted spread by less than a penny.⁴ These results differ from those in Hendershott et al. (2011). We attribute this to a combination of the differences in our AT proxies and instrument as well as our inclusion of a more recent sample period. One possible explanation is that the initial increase in AT during its inception has been far larger in terms of effects than subsequent increases. For the variance measures, a one standard deviation increase in our AT proxy results in a decrease in the proportional daily high-low spread of approximately 0.25% and a decrease in realized variance associated with percentage log returns of approximately 0.12 (or equivalently a reduction in realized daily volatility of approximately 0.35%).

From a welfare perspective the magnitude of the effect is important. As mentioned above and further investigated below, if AT amplifies liquidity dynamics then it will increase the liquidity risk in a market. There is ample evidence that liquidity risk is priced and thus this would result in a higher required return for stocks trading in that market and a corresponding higher

³For the purpose of readability we divide the AT variable by 100 to reduce trailing zeros after the decimal.

⁴It should be noted that this is technically impossible.

cost of capital. Because of this, any benefit in terms of increased liquidity on average, needs to be evaluated against the costs associated with increased variation.

The coefficients on the control variables are also generally in line with what we would expect. The log of market equity is negatively related to both spreads and the high-low price range variable. This is expected given that smaller firms typically have a smaller group of potential investors and are likely to be less liquid. It is also consistent with previous results. We do note that we find a positive and statistically significant relationship between market equity and realized variance. Our market equity variable may be picking up persistence in volatility since we do not consider a dynamic panel setting in this empirical exercise and thus do not control for persistence in the variance term. Although ideally we would include the lag of realized variance as a regressor, as mentioned above, we attempt to avoid a dynamic panel setting. We proxy for this using the lagged value of the price range variable, but we note that this may not fully compensate for the exclusion. The remaining results for the control variables fit with prior research and theory. Higher share turnover, smaller inverse prices (i.e., higher prices) and lower variance all increase liquidity.

Tables 6 through 9 present the results when we allow the parameter to jump discretely over time. The coefficient estimates in Tables 6 through 9 represent the size of the estimated jump in the coefficient and a Chow-type test detailed in Appendix B. Figures 4 through 6 plot both the estimated post-SAW coefficients and the results from period by period cross-sectional regressions.

Our first main finding is that the effect of AT on our measures of market quality is constant for the majority of the sample (the period prior to the financial crisis). This was a relatively placid period for equity markets, and the lack of time varying effects accords with our prior belief that structural breaks in the marginal effect are likely to occur during periods of turmoil. Of note is that the estimated coefficients of the marginal effect of AT on the two spread measures (PES and PQS) over this stable period are positive, however they are both quite small in terms of economic magnitude and statistically insignificant (nor reported here).

The 2007-2008 period covers the financial crisis, a time during which liquidity in many markets tightened substantially. During the financial crisis period we find significant evidence of both positive and negative jumps in the coefficient on AT. For the two spread measures we find evidence of two large positive jumps in the coefficients in April and September/October of 2008 and other smaller jumps around those two time periods. A positive jump in the coefficient represents a reduction in the benefit of AT on spreads and, potentially, a reversal in its effects on spreads. Such is the case for the two large positive jumps mentioned above. We find that during these two months increases in AT lead to an increase in spreads and thus transacting in the

securities with high AT is, other things being equal, costlier than in low AT securities. April and September/October of 2008 represent two particularly volatile periods for equity markets (and markets in general) in the US. In April markets were still rebounding from the bailout of Bear Stearns and its eventual sale to JP Morgan. These events occurred during a period when the exposure of many banks to US housing markets through various structured financial products was beginning to be recognized by investors. Similarly, the failure of Lehman Brothers in September was another event that disquieted the financial markets.

The results for our variance measures are similar, as we also find evidence of both positive and negative jumps during the 2007-2008 period. Of note is that for realized variance we find the jumps in general to be beneficial for investors. That is, we find that increases in AT cause a larger reduction in realized variance. As mentioned above, some caution should be taken with respect to the interpretation of these results due to the fact that variance is generally found to be strongly auto-correlated.

A potential explanation for the variation in the marginal effect of AT is the presence of increased uncertainty. From both a valuation and a regulatory/policy perspective, the periods following large, unpredictable shocks to asset markets can be associated with heightened uncertainty among investors. If investors fear that algorithmic traders possess an informational advantage then it would be precisely during these periods when an increase in AT would cause investors to be most at risk. Although a model of the dynamic effects of AT and uncertainty is beyond the scope of this paper, our results clearly point to a time varying relationship between the effects of AT on various measures of market quality.

Dependent Variable		\hat{AT}_{it}	$\ln(ME)_{it-1}$	T/O_{it-1}	$1/P_{it-1}$	$H-L_{it-1}$	RV_{it-1}
PQS_{it}	Coef.	-0.013	-0.003	0.027	0.619	0.002	
	<i>t</i> -value	-3.61	-2.65	0.73	15.38	5.67	
PES_{it}	Coef.	-0.006	-0.001	-0.077	0.517	0.004	
	<i>t</i> -value	-3.62	-1.83	-3.42	18.51	16.96	
RV_{it}	Coef.	-0.691	0.046	-1.575	7.25	0.415	
	<i>t</i> -value	-12.73	2.39	-2.45	10.19	44.46	
$H-L_{it}$	Coef.	-1.404	-0.038	-6.15	7.88		1.151
	<i>t</i> -value	-11.6	-1.04	-4.71	6.04		35.78

N=378 T=71

This table shows the results of the 2SLS panel regression of our measures of market quality on our AT proxy. The dependent variables are proportional quoted spread, proportional effective spread, daily high-low price range and daily realized variance. In addition to AT, additional regressors included as control variables are the previous month's log of market Cap ($\ln(ME)$), share turnover (T/O), inverse price ($1/P$) and high-low price range ($H-L$). When the dependent variable is the current month's high-low price range, last month's value of realized variance (RV) is used instead to avoid a dynamic panel model. Standard errors are corrected for heteroskedasticity.

Table 5: Instrumental variable panel data model with constant parameters

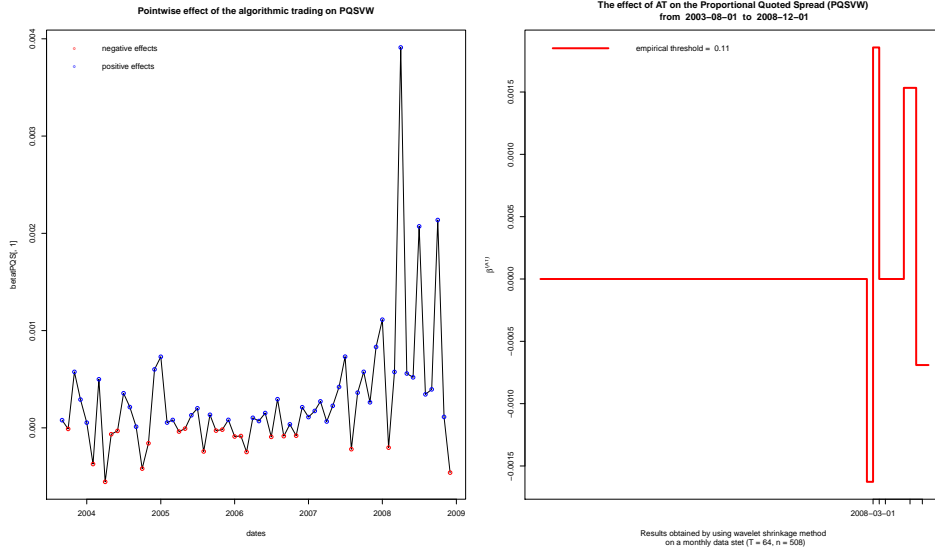


Figure 3: Time varying effect of algorithmic trading on the proportional quoted spread.

	Coef.	Z-value on the difference	p-value	
from 2003-09-01 to 2008-02-01	6.49e-05	-	-	-
from 2008-03-01 to 2008-03-01	6.51e-04	1.650	0.0998	.
from 2008-04-01 to 2008-04-01	4.13e-03	6.420	1.36e-10	***
from 2008-05-01 to 2008-08-01	7.66e-04	-7.420	1.16e-13	***
from 2008-09-01 to 2008-10-01	1.03e-03	0.932	0.3510	
from 2008-11-01 to 2008-12-01	-1.46e-04	-4.620	3.83e-06	***

This table presents the Post-SAW estimates for the parameters and the results of tests for jump significance for the coefficient of AT when the dependent variable is PQS . The column labeled Coef. is the Post-SAW estimate for the parameter and the Z statistic represents a test of the significance of the change from the previous time period (set equal to 0 for the first period). All tests are asymptotic. *** denotes significance at the 0.1% level, ** denotes significance at the 1% level, * denotes significance at the 5% level and . denotes significance at the 10% level.

Table 6: Post-wavelet estimates for the proportional quoted spread.

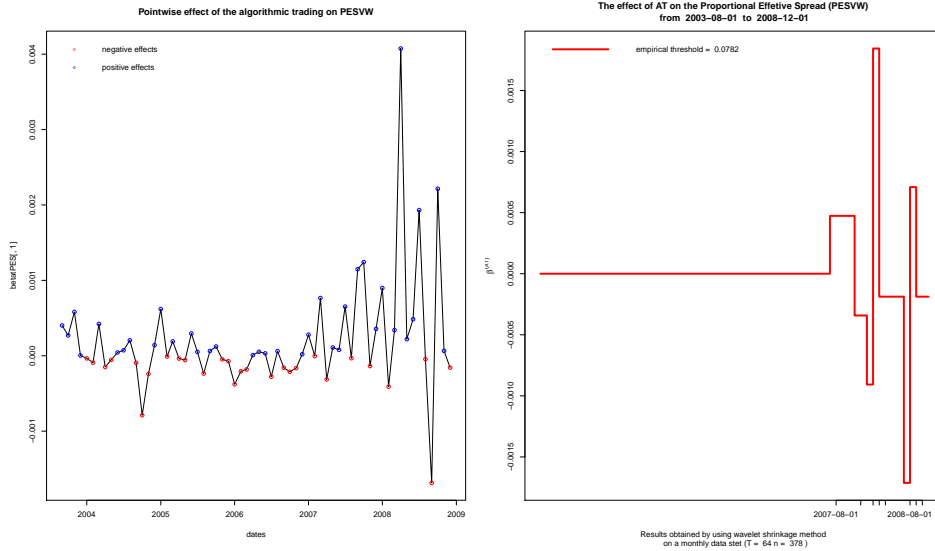


Figure 4: Time varying effect of algorithmic trading on the proportional effective spread.

	Coef.	Z-value on the difference	p-value	
from 2003-09-01 to 2007-08-01	9.06e-06	-	-	
from 2007-09-01 to 2007-12-01	6.73e-04	3.750	0.000176	***
from 2008-01-01 to 2008-02-01	1.35e-04	-2.000	0.045800	*
from 2008-03-01 to 2008-03-01	5.31e-04	1.160	0.248000	
from 2008-04-01 to 2008-04-01	4.19e-03	10.700	< 2.2e-16	***
from 2008-05-01 to 2008-08-01	4.15e-04	-15.600	< 2.2e-16	***
from 2008-09-01 to 2008-09-01	-1.74e-03	-7.540	4.77e-14	***
from 2008-10-01 to 2008-10-01	1.79e-03	11.700	< 2.2e-16	***
from 2008-11-01 to 2008-12-01	3.55e-06	-9.610	< 2.2e-16	***

This table presents the Post-SAW estimates for the parameters and the results of tests for jump significance for the coefficient of AT when the dependent variable is *PES*. The column labeled Coef. is the Post-SAW estimate for the parameter and the Z statistic represents a test of the significance of the change from the previous time period (set equal to 0 for the first period). All tests are asymptotic. *** denotes significance at the 0.1% level, ** denotes significance at the 1% level, * denotes significance at the 5% level and . denotes significance at the 10% level.

Table 7: Post-wavelet estimates for the proportional effective spread.

	Coef.	Z-value on the difference	p-value	
from 2003-09-01 to 2007-06-01	-0.017100	-	-	-
from 2007-07-01 to 2007-07-01	-0.048500	-1.08	0.28200	
from 2007-08-01 to 2007-08-01	-0.154000	-2.68	0.00745	**
from 2007-09-01 to 2008-08-01	-0.012400	5.23	1.65e-07	***
from 2008-09-01 to 2008-09-01	-0.107000	-4.40	1.07e-05	***
from 2008-10-01 to 2008-10-01	-0.000913	4.59	4.33e-06	***
from 2008-11-01 to 2008-12-01	-0.021400	-1.60	0.11000	

This table presents the Post-SAW estimates for the parameters and the results of tests for jump significance for the coefficient of AT when the dependent variable is $H - L$.

The column labeled Coef. is the Post-SAW estimate for the parameter and the Z statistic represents a test of the significance of the change from the previous time period (set equal to 0 for the first period). All tests are asymptotic. *** denotes significance at the 0.1% level, ** denotes significance at the 1% level, * denotes significance at the 5% level and . denotes significance at the 10% level.

Table 8: Post-wavelet estimates for the daily high-low price range.

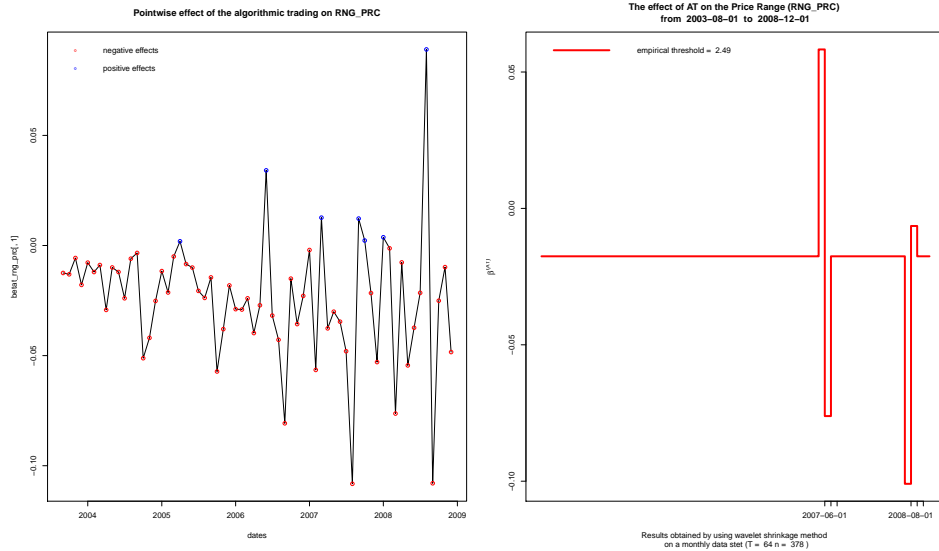


Figure 5: Time varying effect of algorithmic trading on the daily high-low price range.

Table 9: Post-wavelet estimates for the realized variance.

	Coef.	Z-value on the difference	p-value	
from 2003-09-01 to 2008-08-01	-0.008080	-14.20	< 2.2e-16	***
from 2008-09-01 to 2008-09-01	-0.063100	-5.09	3.57e-07	***
from 2008-10-01 to 2008-10-01	0.000888	5.29	1.21e-07	***
from 2008-11-01 to 2008-12-01	-0.007830	-1.26	0.208	

This table presents the Post-SAW estimates for the parameters and the results of tests for jump significance for the coefficient of AT when the dependent variable is RV . The column labeled Coef. is the Post-SAW estimate for the parameter and the Z statistic represents a test of the significance of the change from the previous time period (set equal to 0 for the first period). All tests are asymptotic. *** denotes significance at the 0.1% level, ** denotes significance at the 1% level, * denotes significance at the 5% level and . denotes significance at the 10% level.

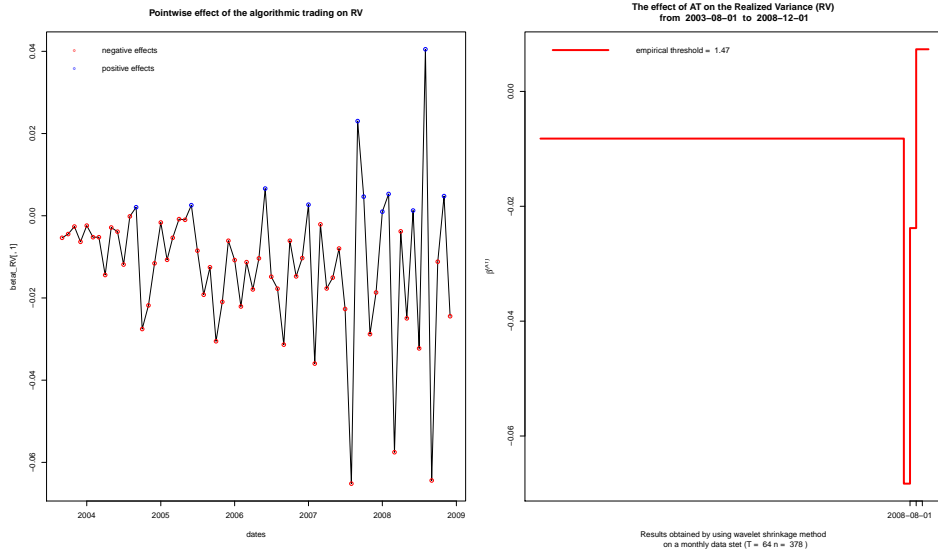


Figure 6: Time varying effect of algorithmic trading on the realized variance.

8 Conclusion

This paper generalizes existing panel model specifications in which the slope parameters are either constant over time or display time heterogeneity. We allow for multiple structural changes that can occur at unknown date points and may affect each slope parameter separately. Consistency under weak forms of dependency and heteroscedasticity in the idiosyncratic errors is established and convergence rates are derived. Our empirical vehicle for highlighting this new methodology addresses the stability of the relationship between Algorithmic Trading (AT) and Market Quality (MQ). We find evidence that the relationship between AT and MQ was disrupted during the time between 2007 and 2008. This period coincides with the beginning of the subprime crisis in the US market and the bankruptcy of the big financial services firm Lehman Brothers.

A Theoretical Results and Proofs

A.1 Proofs of Section 2

Lemma 3 *Let $T = 2^{L-1}$ for some integer $L \geq 2$ and $\beta = (\beta_1, \dots, \beta_T)' \in \mathbb{R}^T$ a vector that possesses exactly one jump at $\tau \in \{1, \dots, T\}$ such that*

$$\beta_t = \begin{cases} \beta_\tau & \text{for } t \in \{1, \dots, \tau\} \\ \beta_{\tau+1} \neq \beta_\tau & \text{for } t \in \{\tau+1, \dots, T\}. \end{cases}$$

Let $w_{lk}(t)$ be defined as (8) and $h_{lk}(t)$ as (9), where $a_{1,1}, a_{l,2k-1}$ and $a_{l,2k}$ are positive real values for all $l \in \{1, \dots, L\}$, and $k \in \{1, \dots, K_l\}$. There then exists unique l_τ non-zero coefficients $\{b_{lk_l} | l_\tau \leq L\}$, where $k_l \in \{1, \dots, K_l\}$, such that

$$\beta_t = \sum_{l=1}^{l_\tau} w_{lk_l}(t) b_{lk_l}.$$

Proof of Lemma 3: To prove the proposition, we show that β_t can be reconstructed by using at most L wavelet basis if it processes exactly one jump, say at $\tau \in \{1, \dots, T\}$. To simplify the exposition, we re-define the wavelet basis $w_{l,k}(t)$, for $l > 1$ as follows:

$$w_{l,k}(t) = a_{l,2k-1}^* h_{l,2k-1}^*(t) - a_{l,2k}^* h_{l,2k}^*(t),$$

where

$$h_{l,k}^*(t) = \begin{cases} 1 & \text{for } t \in \{(2^{L-l-1}(k-1)+1), \dots, (2^{L-l-1}k)\} \\ 0 & \text{else.} \end{cases}$$

This is equivalent to (8). The unique difference is that the coefficients $a_{l,2k-1}^*$ and $a_{l,2k}^*$ are scaled by $\sqrt{2^l}$ in order to simplify the construction of $h_{l,k}^*(t)$ and let it be either 1 or 0.

Note that by construction, there exists a unique $l_\tau \in \{2, \dots, L\}$ and a unique $k_{l_\tau} \in \{1, \dots, 2^{l_\tau-2}\}$ such that

$$w_{l_\tau k_{l_\tau}}(\tau) = a_{l_\tau, 2k_{l_\tau}-1}^* \quad \text{and} \quad w_{l_\tau k_{l_\tau}}(\tau+1) = -a_{l_\tau, 2k_{l_\tau}}^*.$$

Moreover, there exists in each level $l \in \{1, \dots, L | l < l_\tau\}$ at most one basis $w_{lk_l}(t)$ that satisfies the following condition:

$$w_{lk_l}(\tau) = w_{lk_l}(\tau+1) \neq 0.$$

Define the time interval \mathcal{I}_l , for each $l = 1, \dots, l_\tau$, as follows:

$$\mathcal{I}_l = \{t \in \{1, \dots, T\} | w_{lk_l}(t) \neq 0\}.$$

such that

$$\bigcup_{l=1}^{l_\tau} \mathcal{I}_l = \{1, \dots, T\}$$

and

$$\mathcal{I}_{l_\tau} \subset \mathcal{I}_{l_\tau-1} \subset \dots \subset \mathcal{I}_2 \subseteq \mathcal{I}_1 = \{1, \dots, T\}.$$

We now begin with the thinnest interval \mathcal{I}_{l_τ} that contains the jump. Define

$$\beta_t^{(l_\tau)} = \begin{cases} \beta_t = \beta_\tau & \text{if } t \leq \tau \text{ and } t \in \mathcal{I}_{l_\tau} \cap \{t | t \leq \tau\} \\ \beta_t = \beta_{\tau+1} & \text{if } t > \tau \text{ and } t \in \mathcal{I}_{l_\tau} \cap \{t | t > \tau\} \\ 0 & \text{else.} \end{cases}$$

Because $\beta_\tau \neq \beta_{\tau+1}$ and $a_{l_\tau, 2k_{l_\tau}-1}^*, a_{l_\tau, 2k_{l_\tau}}^* > 0$, there exists a non-zero coefficient $b_{l_\tau, k_{l_\tau}} = \frac{\beta_\tau - \beta_{\tau+1}}{a_{l_\tau, 2k_{l_\tau}-1}^* + a_{l_\tau, 2k_{l_\tau}}^*}$ and a constant $\beta^{(l_\tau)} \neq \{\beta_\tau, \beta_{\tau+1}\}$ such that

$$\beta_t^{(l_\tau)} = \begin{cases} \beta_t = \beta^{(l_\tau)} + a_{l_\tau, 2k_{l_\tau}-1}^* b_{l_\tau, k_{l_\tau}} & \text{if } t \leq \tau \text{ and } t \in \mathcal{I}_{l_\tau} \\ \beta_{\tau+1} = \beta^{(l_\tau)} - a_{l_\tau, 2k_{l_\tau}}^* b_{l_\tau, k_{l_\tau}} & \text{if } t > \tau \text{ and } t \in \mathcal{I}_{l_\tau} \\ 0 & \text{else.} \end{cases} \quad (59)$$

Using the definition of $w_{lk}(t)$, we can rewrite (59) as

$$\beta_t^{(l_\tau)} = \begin{cases} \beta_t = \beta^{(l_\tau)} + w_{l_\tau, k_{l_\tau}}(t) b_{l_\tau, k_{l_\tau}} & \text{if } t \in \mathcal{I}_{l_\tau} \\ 0 & \text{else.} \end{cases} \quad (60)$$

Consider the second thinnest interval $\mathcal{I}_{l_\tau-1}$. Let

$$\beta_t^{(l_\tau-1)} = \begin{cases} \beta_t & \text{if } t \in \mathcal{I}_{l_\tau-1} \setminus \mathcal{I}_{l_\tau} \\ \beta^{(l_\tau)} & \text{if } t \in \mathcal{I}_{l_\tau} \\ 0 & \text{else.} \end{cases}$$

Note that β_t is constant over $\mathcal{I}_{l_\tau-1} \setminus \mathcal{I}_{l_\tau}$; it can be either β_τ or $\beta_{\tau+1}$. Now, because $\beta^{(l_\tau)} \neq \{\beta_\tau, \beta_{\tau+1}\}$, we can determine a second unique non-zero coefficient $b_{l_\tau-1, k_{l_\tau-1}}$ and a second unique constant $\beta^{(l_\tau-1)} \neq \{\beta_\tau, \beta_{\tau+1}\}$ such that

$$\beta_t^{(l_\tau-1)} = \begin{cases} \beta^{(l_\tau-1)} + w_{l_\tau-1, k_{l_\tau-1}}(t) b_{l_\tau-1, k_{l_\tau-1}} = \beta_t & \text{if } t \in \mathcal{I}_{l_\tau-1} \setminus \mathcal{I}_{l_\tau} \\ \beta^{(l_\tau-1)} + w_{l_\tau-1, k_{l_\tau-1}}(t) b_{l_\tau-1, k_{l_\tau-1}} = \beta^{(l_\tau)} & \text{if } t \in \mathcal{I}_{l_\tau} \\ 0 & \text{else.} \end{cases}$$

Because $w_{l_\tau, k_{l_\tau}}(t) = 0$ for all $t \notin \mathcal{I}_{l_\tau-1}$ and all $t \in \mathcal{I}_{l_\tau-1} \setminus \mathcal{I}_{l_\tau}$, adding $w_{l_\tau, k_{l_\tau}}(t) b_{l_\tau, k_{l_\tau}}$ on both sides, gives

$$\beta_t^{(l_\tau-1)} + w_{l_\tau, k_{l_\tau}}(t) b_{l_\tau, k_{l_\tau}} = \begin{cases} \beta_t + w_{l_\tau, k_{l_\tau}}(t) b_{l_\tau, k_{l_\tau}} & \text{if } t \in \mathcal{I}_{l_\tau-1} \setminus \mathcal{I}_{l_\tau} \\ \beta^{(l_\tau)} + w_{l_\tau, k_{l_\tau}}(t) b_{l_\tau, k_{l_\tau}} & \text{if } t \in \mathcal{I}_{l_\tau} \\ 0 & \text{else.} \end{cases}$$

Moreover, because $\beta^{(l_\tau)} + w_{l_\tau, k_{l_\tau}}(t)b_{l, k_l} = \beta_t$ for all $t \in \mathcal{I}_{l_\tau}$, we can write

$$\beta_t^{(l_\tau-1)} + w_{l_\tau, k_{l_\tau}}(t)b_{l, k_l} = \begin{cases} \beta^{(l_\tau-1)} + \sum_{l=l_\tau-1}^{l_\tau} w_{l, k_l}(t)b_{l, k_l} = \beta_t & \text{if } t \in \mathcal{I}_{l_\tau-1} \\ 0 & \text{else.} \end{cases}$$

Replacing $\beta_t^{(l_\tau-1)}$ by $\beta_t^{(l_\tau-2)}$ and proceeding with the recursion until $\beta_t^{(l_\tau-l)}$, for $l \in \{2, \dots, l_\tau\}$, we end up with

$$\beta_t^{(l_\tau-l)} + w_{l_\tau-l+1, k_{l_\tau-l+1}}(t)b_{l_\tau-l+1, k_{l_\tau-l+1}} = \begin{cases} \beta^{(l_\tau-l)} + \sum_{s=l_\tau-l}^{l_\tau} w_{s, k_s}(t)b_{s, k_s} = \beta_t & \text{if } t \in \mathcal{I}_{l_\tau-l} \\ 0 & \text{else.} \end{cases} \quad (61)$$

where $\beta^{(l_\tau-l)}$ is constant over $\mathcal{I}_{l_\tau-l}$. Finally, from (61), we can infer that, for all $t \in \mathcal{I}_1 = \{1, \dots, T\}$,

$$\beta_t = \beta^{(1)} + \sum_{l=2}^{l_\tau} w_{l, k_l}(t)b_{l, k_l} \quad \forall t \in \{1, \dots, T\}.$$

Because $\beta^{(1)}$ is a constant and $w_{11}(t) = a_{11} \neq 0$, $\forall t \in \{1, \dots, T\}$, we can express β_t in terms of $l_\tau \leq L$ basis such that

$$\beta_t = \sum_{l=1}^{l_\tau} w_{l, k_l}(t)b_{l, k_l} \quad \forall t \in \{1, \dots, T\}.$$

This completes the proof. \square

Proof of Proposition 1: To prove the assertion, we expand the original vector in a series of S vectors so that each new vector contains only one jump, and make use of Proposition 3. Let β be a $T \times 1$ vector such that

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{\tau_1} \\ \beta_{\tau_1+1} \\ \vdots \\ \beta_{\tau_2} \\ \beta_{\tau_2+1} \\ \vdots \\ \beta_{\tau_S+1} \\ \vdots \\ \beta_T \end{pmatrix} = \begin{pmatrix} \beta_{\tau_1} \\ \beta_{\tau_1} \\ \vdots \\ \beta_{\tau_1} \\ \beta_{\tau_2} \\ \vdots \\ \beta_{\tau_2} \\ \beta_{\tau_3} \\ \vdots \\ \beta_{\tau_S+1} \\ \vdots \\ \beta_{\tau_S+1} \end{pmatrix}$$

where $\{\tau_s \in \{1, \dots, T\} | \tau_1 < \dots < \tau_S\}$. We can transform β in a series of $S + 1$ Vectors, $\bar{\beta}_{\tau_1}, \dots, \bar{\beta}_{\tau_S}$ as follows:

$$\underbrace{\begin{pmatrix} \beta_{\tau_1} \\ \vdots \\ \beta_{\tau_1} \\ \beta_{\tau_2} \\ \vdots \\ \beta_{\tau_2} \\ \beta_{\tau_3} \\ \vdots \\ \beta_{\tau_S} \\ \beta_{\tau_{S+1}} \\ \vdots \\ \beta_{\tau_{S+1}} \end{pmatrix}}_{\beta} = \underbrace{\begin{pmatrix} \beta_{\tau_1} - \beta_{\tau_2} \\ \vdots \\ \beta_{\tau_1} - \beta_{\tau_2} \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\bar{\beta}_{\tau_1}} + \dots + \underbrace{\begin{pmatrix} \beta_{\tau_{S-1}} - \beta_{\tau_S} \\ \vdots \\ \beta_{\tau_{S-1}} - \beta_{\tau_S} \\ \beta_{\tau_{S-1}} - \beta_{\tau_S} \\ \vdots \\ \beta_{\tau_{S-1}} - \beta_{\tau_S} \\ \beta_{\tau_{S-1}} - \beta_{\tau_S} \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\bar{\beta}_{\tau_S}} + \underbrace{\begin{pmatrix} \beta_{\tau_S} \\ \vdots \\ \beta_{\tau_S} \\ \beta_{\tau_S} \\ \vdots \\ \beta_{\tau_S} \\ \beta_{\tau_S} \\ \vdots \\ \beta_{\tau_S} \\ \beta_{\tau_{S+1}} \\ \vdots \\ \beta_{\tau_{S+1}} \end{pmatrix}}_{\bar{\beta}_{\tau_{S+1}}},$$

so that each new vector processes exactly one jump (except $\bar{\beta}_{\tau_{S+1}}$, which is constant over all). From Proposition 3, we know that each vector $\bar{\beta}_{\tau_s}$, $s = 1, \dots, S$, has a unique expansion of the form

$$\bar{\beta}_{\tau_s} = \sum_{l=1}^L \sum_{k=1}^{K_l} w_{lk} b_{lk}^{(s)}$$

with at most L non-zero coefficients in $\{b_{lk}^{(s)}\}_{l=1, \dots, L; k=1, \dots, K_l}$, where

$$K_l = \begin{cases} 1 & \text{if } l = 1 \\ 2^{l-2} & \text{if } l = 2, \dots, L. \end{cases}$$

The fact that $\beta = \sum_{s=1}^{S+1} \bar{\beta}_{\tau_s}$ completes the proof. \square

Proposition 3 *If $a_{1,1}, a_{l,2k-1}$ and $a_{l,2k}$ are chosen for each $l \in \{1, \dots, L\}$ and $k \in \{1, \dots, K_l\}$ such that*

- (i) $a_{l,2k-1}^2 \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T X_{it} Z_{it} h_{l,2k-1}^2(t) + a_{l,2k}^2 \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T X_{it} Z_{it} h_{l,2k}^2(t) = 1,$
- (ii) $a_{l,2k-1} \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T X_{it} Z_{it} h_{l,2k-1}^2(t) - a_{l,2k} \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T X_{it} Z_{it} h_{l,2k}^2(t) = 0$
- (iii) $a_{1,1}^2 \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T X_{it} Z_{it} = 1$

then (a) and (b) are satisfied for all $l, l' \in \{1, \dots, L\}$, $k, k' \in \{1, \dots, K_l\}$, and $k', \in \{1, \dots, K_{l'} | l \neq l'\}$, or $k, k' \in \{1, \dots, K_l | k \neq k'; l = l'\}$.

Proof of Proposition 3: To prove that (i) – (iii) imply the orthonormality conditions (a) and (b), for all $l, l' \in \{1, \dots, L\}$, $k \in \{1, \dots, K_l\}$, and $k' \in \{1, \dots, K_{l'}\}$, it is sufficient to verify the following three statements:

- **(S.1):** condition (b) holds if $l = l'$ and $k' \neq k$.
- **(S.2):** condition (b) holds if (ii) is satisfied for all $l' < l$, and
- **(S.3):** condition (a) holds if (i) and (iii) are satisfied for all $(l, k) = (l', k')$.

Before checking S.1-S.3, we begin with examining the product $\mathcal{Z}_{l,k,it} \mathcal{X}_{l',k',it}$. If $(l, k) \neq (l', k')$,

$$\begin{aligned}
\mathcal{Z}_{l,k,it} \mathcal{X}_{l',k',it} &= Z_{it,lk} Z_{it,l'k'} \\
&= X_{it} Z_{it} (w_{lk}(t) w_{l'k'}(t)) \\
&= X_{it} Z_{it} (a_{l,2k-1} h_{l,2k-1}(t) - a_{l,2k} h_{l,2k}(t)) (a_{l',2k'-1} h_{l',2k'-1}(t) - a_{l',2k'} h_{l',2k'}(t)) \\
&= X_{it} Z_{it} \left(a_{l,2k-1} a_{l',2k'-1} h_{l,2k-1}(t) h_{l',2k'-1}(t) - a_{l,2k-1} a_{l',2k'} h_{l,2k-1}(t) h_{l',2k'}(t) \right. \\
&\quad \left. - a_{l,2k} a_{l',2k'-1} h_{l,2k}(t) h_{l',2k'-1}(t) + a_{l,2k} a_{l',2k'} h_{l,2k}(t) h_{l',2k'}(t) \right)
\end{aligned}$$

If $(l, k) = (l', k')$,

$$\begin{aligned}
\mathcal{Z}_{l,k,it} \mathcal{X}_{l,k,it} &= X_{it} Z_{it} (w_{lk}(t))^2 \\
&= X_{it} Z_{it} (a_{l,2k-1} h_{l,2k-1}(t) - a_{l,2k} h_{l,2k}(t))^2 \\
&= X_{it} Z_{it} \left(a_{l,2k-1}^2 h_{l,2k-1}^2(t) + a_{l,2k}^2 h_{l,2k}^2(t) - 2a_{l,2k-1} a_{l,2k} \underbrace{h_{l,2k-1}(t) h_{l,2k}(t)}_0 \right) \\
&= X_{it} Z_{it} \left(a_{l,2k-1}^2 h_{l,2k-1}^2(t) + a_{l,2k}^2 h_{l,2k}^2(t) \right), \tag{62}
\end{aligned}$$

The product $h_{l,2k-1}(t) h_{l,2k}(t)$ (in the third line) is zero because $h_{l,2k}(t) = 0$, for all $t \in \{((2k-2)2^{L-l} + 1), \dots, ((2k-1)2^{L-l})\}$, $h_{l,2k-1}(t) = 0$, for all $t \in \{((2k-1)2^{L-l} + 1), \dots, (2k2^{L-l})\}$ and both $h_{l,2k}(t) = h_{l,2k-1}(t) = 0$ else.

Consider **(S.1)**. If $l = l'$, and $k' \neq k$, we have, for all $t \in \{1, \dots, T\}$,

$$\begin{aligned}
\mathcal{Z}_{l,k,it} \mathcal{X}_{l',k',it} &= Z_{it} X_{it} \left(a_{l,2k-1} a_{l,2k'-1} \underbrace{h_{l,2k-1}(t) h_{l,2k'-1}(t)}_{=0} - a_{l,2k-1} a_{l,2k'} \underbrace{h_{l,2k-1}(t) h_{l,2k'}(t)}_{=0} \right. \\
&\quad \left. - a_{l,2k} a_{l,2k'-1} \underbrace{h_{l,2k}(t) h_{l,2k'-1}(t)}_{=0} + a_{l,2k} a_{l,2k'} \underbrace{h_{l,2k}(t) h_{l,2k'}(t)}_{=0} \right) \\
&= 0
\end{aligned}$$

This implies (b), for all $l, l' \in \{2, \dots, L | l = l'\}$ and $k, k' \in \{1, \dots, 2^{l-2} | k' \neq k\}$.

Consider **(S.2)**. If $l' < l$, we have by construction either

$$\begin{aligned}\mathcal{Z}_{l,k,it}\mathcal{X}_{l',k',it} &= Z_{it}X_{it}a_{l',2k'}h_{l',2k'}(t)(a_{l,2k-1}h_{l,2k-1}(t) - a_{l,2k}h_{l,2k}(t)) \\ &= a_{l',2k'}\left(Z_{it}X_{it}a_{l,2k-1}h_{l,2k-1}(t)h_{l',2k'}(t) - Z_{it}X_{it}a_{l,2k}h_{l,2k}(t)h_{l',2k'}(t)\right)\end{aligned}$$

or

$$\begin{aligned}\mathcal{Z}_{l,k,it}\mathcal{X}_{l',k',it} &= Z_{it}X_{it}a_{l',2k'-1}h_{l',2k'-1}(t)(a_{l,2k-1}h_{l,2k-1}(t) - a_{l,2k}h_{l,2k}(t)) \\ &= a_{l',2k'-1}\left(Z_{it}X_{it}a_{l,2k-1}h_{l,2k-1}(t)h_{l',2k'-1}(t) - Z_{it}X_{it}a_{l,2k}h_{l,2k}(t)h_{l',2k'-1}(t)\right),\end{aligned}$$

If $h_{l',2k'}(t) = \sqrt{2^l}$, then $h_{l',2k'-1}(t) = 0$ and if $h_{l',2k'-1}(t) = \sqrt{2^l}$, then $h_{l',2k'}(t) = 0$, otherwise both $h_{l',2k'}(t)$ and $h_{l',2k'-1}(t)$ are zeros. Thus condition (ii) ensures (b).

Consider **(S.3)**. From (62), we can easily verify that (a) is a direct result of (i) for all $l \in \{2, \dots, L\}$ and $k \in \{1, \dots, K_l\}$. \square

A.2 Proofs of Section 3

Proof of Lemma 1: The IV estimator of our (modified) wavelets coefficients is given by

$$\begin{aligned}\tilde{b}_{l,k,p} &= \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \mathcal{Z}_{lk,it,p} \Delta y_{it}, \\ &= \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \mathcal{Z}_{lk,it,p} \left(\sum_{l=1}^L \sum_{k=1}^{K_l} \sum_{q=1}^P \mathcal{Z}_{lk,it,q} b_{l,k,q} + \Delta e_{it} \right), \\ &= b_{l,k,p} + \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \sum_{q=1}^P \mathcal{Z}_{lk,it,p} \Delta e_{it}.\end{aligned}$$

The last equality is due to the orthonormality conditions (A) and (B). Subtracting $b_{l,k,p}$ from both sides and multiplying by $\sqrt{n(T-1)}$, we get, for

$l > 1$,

$$\begin{aligned}
\sqrt{n(T-1)}(\tilde{b}_{l,k,p} - b_{l,k,p}) &= \frac{1}{\sqrt{n(T-1)}} \sum_{i=1}^n \sum_{t=2}^T \sum_{q=1}^P \mathcal{Z}_{lk,it,q} \Delta e_{it}, \\
&= \frac{1}{\sqrt{n(T-1)}} \sum_{i=1}^n \sum_{t=2}^T \sum_{q=1}^P W_{lk,pq}(t) Z_{it,q} \Delta e_{it}, \\
&= \frac{1}{\sqrt{n(T-1)}} \sum_{i=1}^n \sum_{t=2}^T \sum_{q=1}^P A_{l,2k,pq} h_{l,2k}(t) Z_{it,q} \Delta e_{it} \\
&\quad - \frac{1}{\sqrt{n(T-1)}} \sum_{i=1}^n \sum_{t=2}^T \sum_{q=1}^P A_{l,2k-1,pq} h_{l,2k-1}(t) Z_{it,q} \Delta e_{it}, \\
&= \frac{1}{\sqrt{n(2^{L-l-1}-1)}} \sum_{q=1}^P A_{l,2k,pq} \sum_{i=1}^n \sum_{t \in \{h_{l,2k}(t) \neq 0\}} Z_{it,q} \Delta e_{it} \\
&\quad - \frac{1}{\sqrt{n(2^{L-l-1}-1)}} \sum_{q=1}^P A_{l,2k-1,pq} \sum_{i=1}^n \sum_{t \in \{h_{l,2k-1}(t) \neq 0\}} Z_{it,q} \Delta e_{it},
\end{aligned}$$

where $W_{lk,pq}(t)$ and $A_{l,m,pq}$ are the (p, q) - elements of the matrices $W_{l,k}(t)$ and $A_{l,m}$, respectively. and, for $l = 1$,

$$\sqrt{n(T-1)}(\tilde{b}_{1,1,p} - b_{1,1,p}) = \frac{1}{\sqrt{n(2^L-1)}} \sum_{q=1}^P A_{1,1,pq} \sum_{i=1}^n \sum_{t=2}^T Z_{it,q} \Delta e_{it}.$$

By Assumption B.(i), we know that $E_c(Z_{it} \Delta e_{it}) = 0$, for all i and t . The law of total expectation implies

$$E(\sqrt{n(T-1)}(\tilde{b}_{l,k,p} - b_{l,k,p})) = 0,$$

for all l and k . The total variance, for $l > 1$, can be written as

$$\begin{aligned}
\Sigma_{l,k,p} &= E((\sqrt{n(T-1)}(\tilde{b}_{l,k,p} - b_{l,k,p}))^2), \\
&= E\left(\frac{1}{n(2^{L-l-1}-1)} \sum_{q,r=1}^P A_{l,2k,pq} A_{l,2k,pr} \sum_{i,j=1}^n \sum_{t,s \in H} Z_{it,q} Z_{js,r} E_c(\Delta e_{it} \Delta e_{js})\right) \\
&\quad + E\left(\frac{1}{n(2^{L-l-1}-1)} \sum_{q,r=1}^P A_{l,2k-1,pq} A_{l,2k-1,pr} \sum_{i,j=1}^n \sum_{t,s \in H} Z_{it,q} Z_{js,r} E_c(\Delta e_{it} \Delta e_{js})\right), \\
&= \Pi_{l,k,1} + \Pi_{l,k,2},
\end{aligned}$$

where $\sum_{q,r=1}^P$, $\sum_{i,j=1}^n$ and $\sum_{t,s \in H}$ denote the double summations $\sum_{q=1}^P \sum_{r=1}^P$, $\sum_{i=1}^n \sum_{j=1}^n$ and $\sum_{t \in \{H_{l,2k}(t) \neq 0\}} \sum_{s \in \{H_{l,2k}(s) \neq 0\}}$, respectively.

For $l = 1$,

$$\begin{aligned} \Sigma_{1,1,p} &:= E((\sqrt{n(T-1)}(\tilde{b}_{1,1,p} - b_{1,1,p}))^2) \\ &= E\left(\sum_{q,r=1}^P \frac{1}{n(2^L-1)} A_{1,1,pq} A_{1,1,pr} \sum_{i,j=1}^n \sum_{t,s=2}^T Z_{it,q} Z_{js,r} E_c(\Delta e_{it} \Delta e_{js})\right). \end{aligned}$$

By using Assumption C, we can infer

$$\begin{aligned} \Pi_{l,k,1} &= E\left(\frac{1}{n(2^{L-l-1}-1)} \sum_{q,r=1}^P A_{l,2k,pq} A_{l,2k,pr} \sum_{i,j=1}^n \sum_{t,s \in H} Z_{it,q} Z_{js,r} \sigma_{ij,ts}\right), \\ &\leq E\left(\frac{1}{n(2^{L-l-1}-1)} \sum_{q,r=1}^P A_{l,2k,pq} A_{l,2k,pr} \sum_{i,j=1}^n \sum_{t,s \in H} Z_{it,q} Z_{js,r} |\sigma_{ij,ts}|\right), \\ &\leq E\left(A_{l,2k,pp}^2 \frac{1}{n(2^{L-l-1}-1)} \sum_{i,j=1}^n \sum_{t,s \in \{h_{l,2k}(t), h_{l,2k}(s) \neq 0\}} Z_{it,p} Z_{js,p}\right) \bar{\sigma}, \\ \Pi_{l,k,2} &\leq E\left(\frac{1}{n(2^{L-l-1}-1)} \sum_{q,r=1}^P A_{l,2k-1,pq} A_{l,2k-1,pr} \sum_{i,j=1}^n \sum_{t,s \in H} Z_{it,q} Z_{js,r} |\sigma_{ij,ts}|\right), \text{ and} \\ \Sigma_{1,1,p} &\leq E\left(\frac{1}{n(2^L-1)} \sum_{q,r=1}^P A_{1,1,pq} A_{1,1,pr} \sum_{i,j=1}^n \sum_{t,s=2}^T Z_{it,q} Z_{js,r} |\sigma_{ij,ts}|\right). \end{aligned}$$

Because $E(\|A_{l,2k}\|^4)$ and $E(\|A_{l,2k-1}\|^4)$ are bounded uniformly in l, k , and $E(\|Z_{it}\|^4)$, and $|\sigma_{ij,ts}|$ is bounded uniformly in i, j, t, s (see Assumptions B and C), we can easily show (by Cauchy-Schwarz inequality) that $\Sigma_{l,k,p} \leq M$ is bounded uniformly in l, k, p . Using Assumption B(iii), we can write

$$\begin{aligned} P\left(\left|\tilde{b}_{l,k,p} - b_{l,k,p}\right| > M^{\frac{1}{2}} \frac{c}{\sqrt{n(T-1)}}\right) &\leq P\left(\left|\Sigma_{l,k,p}^{-\frac{1}{2}} \sqrt{n(T-1)} \left|\tilde{b}_{l,k,p} - b_{l,k,p}\right|\right| > c\right), \\ &\leq \frac{1}{c} \exp(-\frac{c^2}{2}). \end{aligned} \quad (63)$$

Using Boole's inequality and (63), we get

$$\begin{aligned} P\left(\sup_{l,k,p} \left|\tilde{b}_{l,k,p} - b_{l,k,p}\right| > M^{\frac{1}{2}} \frac{c}{\sqrt{n(T-1)}}\right) &\leq \sum_{l,k,p} P\left(\left|\tilde{b}_{l,k,p} - b_{l,k,p}\right| > M^{\frac{1}{2}} \frac{c}{\sqrt{n(T-1)}}\right), \\ &\leq (2^{L-1} \underline{P}) \frac{1}{c} \exp(-\frac{c^2}{2}), \\ &= (T-1) \underline{P} \frac{1}{c} \exp(-\frac{c^2}{2}), \end{aligned}$$

where $\sum_{l,k,p}$ denotes the triple summation $\sum_{l=1}^L \sum_{k=1}^{K_l} \sum_{p=1}^P$. The assertion of the theorem follows by replacing c with $\sqrt{2 \log((T-1)P)} c^*$ for any $c^* > 0$. \square

Proof of Theorem 1: We have first to prove that (i) : $\sup_t |\tilde{\gamma}_{t,p} - \gamma_{t,p}| = o_p(1)$ for all $p \in \{1, \dots, P\}$ if $\sqrt{T-1} \lambda_{n,T} \rightarrow 0$, as $n, T \rightarrow \infty$ or $n \rightarrow \infty$ and T is fixed, and then conclude that (ii) : $\frac{1}{T-1} \sum_{t=2}^T \|\tilde{\gamma}_t - \gamma_t\|^2 = O_p((\log(T-1)/n)^\kappa)$, if $\sqrt{T-1} \lambda_{n,T} \sim (\log(T-1)/n)^{\kappa/2}$, for $\kappa \in]0, 1]$.

By construction,

$$\tilde{\gamma}_{t,p} - \gamma_{t,p} = \sum_{q=1}^P \sum_{l=1}^L \sum_{k=1}^{K_l} W_{lk,pq}(t) \hat{b}_{l,k,q} - \sum_{q=1}^P \sum_{l=1}^L \sum_{k=1}^{K_l} W_{lk,pq}(t) b_{l,k,q} \quad (64)$$

where

$$\hat{b}_{l,k,q} = \tilde{b}_{l,k,q} - \tilde{b}_{l,k,q} \mathbf{I}(|\tilde{b}_{l,k,q}| < \lambda_{n,T}). \quad (65)$$

and

$$\begin{aligned} W_{lk,pq}(t) &= A_{l,2k,pq}(t) H_{l,2k}(t) - A_{l,2k-1,pq}(t) H_{l,2k-1}(t), \\ &= \sqrt{2^{l-2}} A_{l,2k,pq} \mathbf{I}(H_{l,2k}(t) \neq 0) - \sqrt{2^{l-2}} A_{l,2k-1,pq} \mathbf{I}(H_{l,2k-1}(t) \neq 0). \end{aligned} \quad (66)$$

Plugging (65) and (66) in (64) and using the absolute value inequality, we get

$$\begin{aligned} |\tilde{\gamma}_{t,p} - \gamma_{t,p}| &\leq \sum_{q=1}^P \sum_{l=1}^L \sum_{k=1}^{K_l} \sqrt{2^{l-2}} |A_{l,2k,pq} \mathbf{I}(H_{l,2k}(t) \neq 0) (\tilde{b}_{l,k,q} - b_{l,k,q})| \\ &\quad + \sum_{q=1}^P \sum_{l=1}^L \sum_{k=1}^{K_l} \sqrt{2^{l-2}} |A_{l,2k,pq} \mathbf{I}(H_{l,2k}(t) \neq 0) \tilde{b}_{l,k,q} \mathbf{I}(|\tilde{b}_{l,k,q}| < \lambda_{n,T})| \\ &\quad + \sum_{q=1}^P \sum_{l=1}^L \sum_{k=1}^{K_l} \sqrt{2^{l-2}} |A_{l,2k-1,pq} \mathbf{I}(H_{l,2k-1}(t) \neq 0) (\tilde{b}_{l,k,q} - b_{l,k,q})| \\ &\quad + \sum_{q=1}^P \sum_{l=1}^L \sum_{k=1}^{K_l} \sqrt{2^{l-2}} |A_{l,2k-1,pq} \mathbf{I}(H_{l,2k-1}(t) \neq 0) \tilde{b}_{l,k,q} \mathbf{I}(|\tilde{b}_{l,k,q}| < \lambda_{n,T})|, \\ &= a + b + c + d. \end{aligned}$$

Because $\tilde{b}_{l,k,p} \mathbf{I}(|\tilde{b}_{l,k,p}| < \lambda_{n,T}) < \lambda_{n,T}$ and $|\tilde{b}_{l,k,p} - b_{l,k,p}| \leq \sup_{l,k,p} |\tilde{b}_{l,k,p} -$

$b_{l,k,p}$ for all $p \in \{1, \dots, \underline{P}\}$, we can write

$$\begin{aligned}
a &\leq \sup_{l,k,p} |\tilde{b}_{l,k,p} - b_{l,k,p}| \sum_{q=1}^{\underline{P}} \sum_{l=1}^L \sum_{k=1}^{K_l} |A_{l,2k,pq} \sqrt{2^{l-2}} \mathbf{I}(H_{l,2k}(t) \neq 0)|, \\
b &\leq \lambda_{n,T} \sum_{q=1}^{\underline{P}} \sum_{l=1}^L \sum_{k=1}^{K_l} |A_{l,2k,pq} \sqrt{2^{l-2}} \mathbf{I}(H_{l,2k}(t) \neq 0)|, \\
c &\leq \sup_{l,k,p} |\tilde{b}_{l,k,p} - b_{l,k,p}| \sum_{q=1}^{\underline{P}} \sum_{l=1}^L \sum_{k=1}^{K_l} |A_{l,2k-1,pq} \sqrt{2^{l-2}} \mathbf{I}(H_{l,2k}(t) \neq 0)|, \text{ and} \\
d &\leq \lambda_{n,T} \sum_{q=1}^{\underline{P}} \sum_{l=1}^L \sum_{k=1}^{K_l} |A_{l,2k-1,pq} \sqrt{2^{l-2}} \mathbf{I}(H_{l,2k-1}(t) \neq 0)|.
\end{aligned}$$

By Assumption B, $E(\|A_{l,2k}\|^4)$ and $E(\|A_{l,2k-1}\|^4)$ are bounded uniformly in l and k . We can deduce that

$$\begin{aligned}
\sum_{q=1}^{\underline{P}} \sum_{l=1}^L \sum_{k=1}^{K_l} |A_{l,2k,pq} \sqrt{2^{l-2}} \mathbf{I}(H_{l,2k}(t) \neq 0)| &= O_p(1) \sum_{l=1}^L \sum_{k=1}^{K_l} |\sqrt{2^{l-2}} \mathbf{I}(H_{l,2k}(t) \neq 0)| \text{ and} \\
\sum_{q=1}^{\underline{P}} \sum_{l=1}^L \sum_{k=1}^{K_l} |A_{l,2k-1,pq} \sqrt{2^{l-2}} \mathbf{I}(H_{l,2k-1}(t) \neq 0)| &= O_p(1) \sum_{l=1}^L \sum_{k=1}^{K_l} |\sqrt{2^{l-2}} \mathbf{I}(H_{l,2k-1}(t) \neq 0)|.
\end{aligned}$$

Moreover, from the construction of $H_{l,2k}(t)$ and $H_{l,2k-1}(t)$, we can easily verify that

$$\sup_t \sum_{l=1}^L \sum_{k=1}^{K_l} \sqrt{2^{l-2}} \mathbf{I}(H_{l,2k-1}(t) \neq 0) = \sum_{l=1}^L \sqrt{2^{l-2}} = O(\sqrt{2^{L-1}}) = O(\sqrt{T-1})$$

By Lemma 1, we can infer that

$$\begin{aligned}
\sup_{t,p} |\tilde{\gamma}_{t,p} - \gamma_{t,p}| &= \sup_{l,k,p} |\tilde{b}_{l,k,p} - b_{l,k,p}| \times O_p(\sqrt{T-1}) + \lambda_{n,T} \times O_p(\sqrt{T-1}), \\
&= O_p\left(\sqrt{\frac{\log(T-1)}{n}} + \sqrt{T-1} \lambda_{n,T}\right). \tag{67}
\end{aligned}$$

Assertion (i) follows immediately if $\sqrt{T-1} \lambda_{n,T} \rightarrow 0$ with $\log(T-1)/n \rightarrow 0$, as $n, T \rightarrow \infty$.

Consider Assertion (ii). Let $\mathcal{L}_p := \{(l, k) | b_{l,k,p} = 0\}$ denote the set of double indexes corresponding to the non-zero true wavelet coefficients so that $\gamma_{t,p} = \sum_{q=1}^{\underline{P}} \sum_{l=1}^L \sum_{k=1}^{K_l} W_{l,k,pq}(t) b_{l,k,q}$ can be written as

$$\gamma_{t,p} = \sum_{q=1}^{\underline{P}} \sum_{(l,k) \in \mathcal{L}_p} W_{l,k,pq}(t) b_{l,k,q},$$

and $\tilde{\gamma}_{t,p} = \sum_{q=1}^P \sum_{l=1}^L \sum_{k=1}^{K_l} W_{lk,pq}(t) \hat{b}_{l,k,q}$ as

$$\tilde{\gamma}_{t,p} = \sum_{q=1}^P \sum_{(l,k) \in \mathcal{L}_p} W_{lk,pq}(t) \hat{b}_{l,k,q} + \sum_{q=1}^P \sum_{(l,k) \notin \mathcal{L}_p} W_{lk,pq}(t) \hat{b}_{l,k,q}.$$

The difference, can be written as

$$\tilde{\gamma}_{t,p} - \gamma_{t,p} = \sum_{q=1}^P \sum_{(l,k) \in \mathcal{L}_p} W_{lk,pq}(t) (\hat{b}_{l,k,q} - b_{l,k,q}) + \sum_{q=1}^P \sum_{(l,k) \notin \mathcal{L}_p} W_{lk,pq}(t) \hat{b}_{l,k,q}.$$

Averaging the square, we get

$$\begin{aligned} \frac{1}{T-1} \sum_{t=2}^{T-1} (\tilde{\gamma}_{t,p} - \gamma_{t,p})^2 &= \frac{1}{T-1} \sum_{t=2}^{T-1} \left(\sum_{q=1}^P \sum_{(l,k) \in \mathcal{L}_p} W_{lk,pq}(t) (\hat{b}_{l,k,q} - b_{l,k,q}) \right)^2 \\ &\quad + \frac{1}{T-1} \sum_{t=2}^{T-1} \left(\sum_{q=1}^P \sum_{(l,k) \notin \mathcal{L}_p} W_{lk,pq}(t) \hat{b}_{l,k,q} \right)^2 \\ &\quad - \frac{1}{T-1} \sum_{t=2}^{T-1} \left(\sum_{q=1}^P \sum_{(l,k) \in \mathcal{L}_p} W_{lk,pq}(t) (\hat{b}_{l,k,q} - b_{l,k,q}) \right) \times \\ &\quad \left(\sum_{q=1}^P \sum_{(l,k) \notin \mathcal{L}_p} W_{lk,pq}(t) \hat{b}_{l,k,q} \right), \\ &= \frac{1}{T-1} \sum_{t=2}^{T-1} e_t^2 + \frac{1}{T-1} \sum_{t=2}^{T-1} f_t^2 - \frac{1}{T-1} \sum_{t=2}^{T-1} e_t f_t. \end{aligned}$$

From the analysis of assertion (i), we can see that

$$\begin{aligned} e_t &= \sup_{l,k,p} |\hat{b}_{l,k,p} - b_{l,k,p}| O_p(1) \sum_{q=1}^P \sum_{(l,k) \in \mathcal{L}_p} \sqrt{2^{l-1}} \mathbf{I}(H_{l,2k-1}(t) \neq 1; H_{l,2k}(t) \neq 1) \\ &= O_p\left(\sqrt{\frac{\log(T-1)}{n(T-1)}} + \lambda_{n,T}\right) \sum_{q=1}^P \sum_{(l,k) \in \mathcal{L}_p} \sqrt{2^{l-1}} \mathbf{I}(H_{l,2k-1}(t) \neq 1; H_{l,2k}(t) \neq 1), \end{aligned}$$

and

$$f_t = \sup_{(l,k) \in \mathcal{L}_{p,p}} |\hat{b}_{l,k,p}| O_p(1) \sum_{q=1}^P \sum_{(l,k) \in \mathcal{L}_p} \sqrt{2^{l-1}} \mathbf{I}(H_{l,2k-1}(t) \neq 1; H_{l,2k}(t) \neq 1).$$

Using Cauchy-Schwarz inequality to $(\sum_{q=1}^P \sum_{(l,k) \in \mathcal{L}_p} \sqrt{2^{l-1}} \mathbf{I}(H_{l,2k-1}(t) \neq 1; H_{l,2k}(t) \neq 1))^2$ over (l, k) , we can infer that

$$e_t^2 \leq O_p\left(\frac{\log(T-1)}{n(T-1)} + \lambda_{n,T}^2\right) \sum_{q=1}^P \sum_{(l,k) \in \mathcal{L}_p} 2^{l-1} \mathbf{I}(H_{l,2k-1}(t) \neq 1; H_{l,2k}(t) \neq 1),$$

and

$$\frac{1}{T-1} \sum_{t=2}^{T-2} f_t^2 \leq \left(\sup_{(l,k) \in \mathcal{L}_{p,p}} |\hat{b}_{l,k,p}| \right)^2 O_p(T-1).$$

If $\sqrt{T-1} \lambda_{n,T} \sim (\log(T-1)/n)^{\kappa/2}$, then $\text{plim}(\frac{1}{T-1} \sum_{t=2}^{T-2} f_t^2) = 0$ as T and/or n pass to infinity, for any $\kappa \in]0, 1[$.

Let us now examine the average of e_t^2 over t . If, in total, the maximal number of jumps is $S^* = \sum_p^P S_p$, then by Proposition 1 the number of non-zero coefficients is at most $(S^* + 1)L$. By taking the average of e_t^2 over t , we can hence infer that

$$\frac{1}{T-1} \sum_{t=2}^{T-1} e_t^2 \leq O_p\left(\frac{\log(T-1)}{n(T-1)} + \lambda_{n,T}^2\right) (\min\{(S^* + 1) \log(T-1), (T-1)\}).$$

Finally, because $\text{plim}(\frac{1}{T-1} \sum_{t=2}^{T-2} f_t^2) = 0$, by Cauchy-Schwarz inequality, we can infer that $\frac{1}{T-1} \sum_{t=2}^{T-1} e_t f_t$ also can be neglected. Thus

$$\frac{1}{T-1} \sum_{t=2}^{T-1} (\tilde{\gamma}_{t,p} - \gamma_{t,p})^2 = O_p\left(\frac{J^* (\log(T-1)/n)^\kappa}{(T-1)}\right),$$

where $J^* = \min\{(S^* + 1) \log(T-1), (T-1)\}$. This completes the proof. \square

A.3 Proofs of Section 4

Proof of Lemma 2: We have to show that

$$\sup_{k,p \in \{1, \dots, P\}} \left| \tilde{c}_{L,k,p}^{(m)} - c_{L,k,p}^{(m)} \right| = O_p\left(\sqrt{\log(T-1)/(n(T-1))}\right),$$

for $m = s, u$.

For $p \in \{1, \dots, P\}$ and $m = s$, we have by construction

$$\begin{aligned} \tilde{c}_{L,k,p}^{(s)} - c_{L,k,p}^{(s)} &= \frac{1}{T-1} \sum_{t=2}^T \psi_{L,k}(t-1) (\tilde{\gamma}_{t,p} - \gamma_{t,p}), \\ &= \frac{1}{T-1} \sum_{t=2}^T \psi_{L,k}(t-1) \sum_{l,m,q} W_{l,m,p,q}(t) (\tilde{b}_{l,m,q} - b_{l,m,q}), \\ &= \frac{1}{T-1} \sum_{t \in \{\psi_{L,k}(t-1) \neq 0\}} \psi_{L,k}(t-1) \sum_{l,m,q} W_{l,m,p,q}(t) (\tilde{b}_{l,m,q} - b_{l,m,q}), \end{aligned}$$

where $\sum_{l,m,q}$ denotes the triple summation $\sum_{l=1}^L \sum_{k=1}^{K_l} \sum_{q=1}^P$.

Taking the absolute value, we obtain

$$|\tilde{c}_{L,k,p}^{(s)} - c_{L,k,p}^{(s)}| \leq \sup_{l,k,p} |\tilde{b}_{l,k,p} - b_{l,k,p}| \frac{1}{T-1} \sum_{t \in \{\psi_{L,k}(t-1) \neq 0\}} \left| \psi_{L,k}(t-1) \sum_{l,m,q} W_{l,m,p,q}(t) \right|.$$

Recall that $\frac{1}{T-1} \sum_{t \in \{\psi_{L,k}(t-1) \neq 0\}} \psi_{L,k}(t-1)^2 = 1$. By using Cauchy-Schwarz inequality, we can easily verify that

$$\frac{1}{T-1} \sum_{t \in \{\psi_{L,k}(t-1) \neq 0\}} \left| \psi_{L,k}(t-1) \sum_{l,m,q} W_{l,m,p,q}(t) \right| \leq \left(\frac{1}{T-1} \sum_{t \in \{\psi_{L,k}(t-1) \neq 0\}} \left(\sum_{l,m,q} W_{l,m,p,q}(t) \right)^2 \right)^{1/2}.$$

Because the support of $\psi_{L,k}(t-1)$ is of length 2 ($\sum_t \mathbf{I}(t \in \{\psi_{L,k}(t-1) \neq 0\}) = 2$), by using a similar analysis to that used in the proof of Theorem 1, we can easily verify that the term in the last inequality is $O_p(1)$. By Lemma 1, we can hence infer that

$$|\tilde{c}_{L,k,p}^{(s)} - c_{L,k,p}^{(s)}| \leq \sup_{l,k,p} |\tilde{b}_{l,k,p} - b_{l,k,p}| O_p(1) = O_p(\sqrt{\log(T-1)/n(T-1)}).$$

The proof of $\sup_{L,k,p} |\tilde{c}_{L,k,p}^{(u)} - c_{L,k,p}^{(u)}|$ being $O_p(\sqrt{\log(T-1)/n(T-1)})$ is similar and thus omitted. \square

Proof of Theorem 2: To prove the assertion, we show, in a first part, that asymptotically no jump can be detected in the stability intervals if $\lambda_{n,T}$ satisfies Condition c.1. In a second part, we show that all existing jumps must be asymptotically identified if $\lambda_{n,T}$ satisfies Condition c.2.

We begin with defining the following sets for each $p \in \{1, \dots, P\}$:

$$\begin{aligned} \mathcal{J}_p &:= \{\tau_{1,p}, \dots, \tau_{S_p,p}\}, \\ \mathcal{J}_p^c &:= \{1, \dots, T\} \setminus \mathcal{J}_p, \\ \overline{\mathcal{J}}_p &:= \{2, 4, \dots, T-1\} \cap \mathcal{J}_p, \\ \underline{\mathcal{J}}_p &:= \{3, 5, \dots, T\} \cap \mathcal{J}_p, \\ \overline{\mathcal{J}}_p^c &:= \{2, 4, \dots, T-1\} \setminus \overline{\mathcal{J}}_p, \text{ and} \\ \underline{\mathcal{J}}_p^c &:= \{3, 5, \dots, T\} \setminus \underline{\mathcal{J}}_p. \end{aligned}$$

Here, \mathcal{J}_p is the set of all jump locations for parameter $\beta_{t,p}$, \mathcal{J}_p^c is its complement, which contains only the stability intervals, $\overline{\mathcal{J}}_p$ is the set of all even jump locations and $\underline{\mathcal{J}}_p$ is the set of all odd jump locations so that $\overline{\mathcal{J}}_p \cap \underline{\mathcal{J}}_p = \emptyset$ and $\overline{\mathcal{J}}_p \cup \underline{\mathcal{J}}_p = \mathcal{J}_p$. Finally, the sets $\overline{\mathcal{J}}_p^c$ and $\underline{\mathcal{J}}_p^c$ define the complements of $\overline{\mathcal{J}}_p$ and $\underline{\mathcal{J}}_p$, respectively.

Define the event

$$\omega_{n,T} := \left\{ \sup_{t \in \mathcal{J}_p^c, p \in \{1, \dots, P\}} \{ |\Delta \tilde{\beta}_{t,p}^{(u)}| \mathbf{I}_{\overline{\mathcal{J}}_p^c} + |\Delta \tilde{\beta}_{t,p}^{(s)}| \mathbf{I}_{\underline{\mathcal{J}}_p^c} \} = 0 \right\},$$

where $\mathbf{I}_{\overline{\mathcal{J}}_p^c} = \mathbf{I}(t \in \overline{\mathcal{J}}_p^c)$, $\mathbf{I}_{\underline{\mathcal{J}}_p^c} = \mathbf{I}(t \in \underline{\mathcal{J}}_p^c)$ and $\mathbf{I}(\cdot)$ is the indicator function.

To prove that no jump can be identified in the stability intervals, we have to show, that $P(\omega_{n,T}) \rightarrow 1$, if $\sqrt{\frac{n(T-1)}{\log(T-1)}} \lambda_{n,T} \rightarrow \infty$, as $n, T \rightarrow \infty$ or as $n \rightarrow \infty$ and T is fixed. Note that $\overline{\mathcal{J}}_p^c$ and $\underline{\mathcal{J}}_p^c$ are adjacent.

Let's now start with the no-jump case in $\overline{\mathcal{J}}_p^c$. By construction, we have, for all $t \in \{2, 4, \dots, T-1\}$,

$$\Delta \tilde{\beta}_{t,p}^{(u)} = \sum_{k=1}^{K_L} \Delta \psi_{L,k}(t) \hat{c}_{L,k,p}^{(u)}$$

Recall that at $l = L$, the construction of the wavelets basis implies that at each $t \in \{2, 4, \dots, T-1\}$ there is only one differenced basis $\Delta \psi_{L,k}(t)$ that is not zero. Let $\mathcal{K}_p^c = \{k | \Delta \psi_{L,k}(t) \neq 0, t \in \overline{\mathcal{J}}_p^c\} = \{k | \Delta \psi_{L,k}(t-1) \neq 0, t \in \underline{\mathcal{J}}_p^c\}$. We can infer that $\{\sup_{t \in \overline{\mathcal{J}}_p^c} |\sum_{k=1}^{K_L} \Delta \psi_{L,k}(t) \hat{c}_{L,k,p}^{(u)}| = 0\}$ occurs only if $\{\sup_{k \in \mathcal{K}_p^c} |\hat{c}_{L,k,p}^{(u)}| = 0\}$ occurs.

By analogy, we can show the same assertion for the complement set $\underline{\mathcal{J}}_p^c$, i.e., $\{\sup_{t \in \underline{\mathcal{J}}_p^c} |\Delta \tilde{\beta}_{t,p}^{(s)}| = 0\}$ occurs only if $\{\sup_{k \in \mathcal{K}_p^c} |\hat{c}_{L,k,p}^{(s)}| = 0\}$ occurs.

To study $P(\omega_{n,T})$, it is hence sufficient to study

$$P\left(\sup_{k \in \mathcal{K}_p^c, m, p \in \{1, \dots, P\}} |\hat{c}_{L,k,p}^{(m)}| = 0\right) = P\left(\sup_{k \in \mathcal{K}_p^c, m, p \in \{1, \dots, P\}} |\tilde{c}_{L,k,p}^{(m)}| < \lambda_{n,T}\right).$$

By Lemma 2, $\sup_{k \in \mathcal{K}_p^c, m, p \in \{1, \dots, P\}} |\tilde{c}_{L,k,p}^{(m)}| = O_p(\sqrt{\log(T-1)/n(T-1)})$, since $\hat{c}_{L,k,p}^{(m)} = 0$, for all $k \in \mathcal{K}_p^c$, and $p \in \{1, \dots, P\}$. Thus, if $\sqrt{\frac{n(T-1)}{\log(T-1)}} \lambda_{n,T} \rightarrow \infty$, as $n, T \rightarrow \infty$ or $n \rightarrow \infty$ and T is fixed, then $P(\omega_{n,T}) \rightarrow 1$.

To complete the proof and demonstrate that all true jumps will be asymptotically identified, we suppose that there exists a jump location $\tau_{j,p} \in \overline{\mathcal{J}}_p \cup \underline{\mathcal{J}}_p$ for at least one $p \in \{1, \dots, P\}$ that is not detected and show the contradiction. If $\tau_{j,p} \in \overline{\mathcal{J}}_p$, then

$$|\Delta \tilde{\beta}_{\tau_{j,p},p}^{(u)}| \mathbf{I}_{\overline{\mathcal{J}}_p} + |\Delta \tilde{\beta}_{\tau_{j,p},p}^{(s)}| \mathbf{I}_{\underline{\mathcal{J}}_p} = |\Delta \tilde{\beta}_{\tau_{j,p},p}^{(u)}|.$$

Adding and subtracting $\Delta \beta_{\tau_{j,p},p}^{(u)}$, we get

$$\begin{aligned} \Delta \tilde{\beta}_{\tau_{j,p},p}^{(u)} &= \sum_{k=1}^{K_L} \Delta \psi_{L,k}(\tau_{j,p}) (\tilde{c}_{L,k,p}^{(u)} - c_{L,k,p}^{(u)}) - \sum_{k=1}^{K_L} \Delta \psi_{L,k}(\tau_{j,p}) \tilde{c}_{L,k,p}^{(u)} \mathbf{I}(\tilde{c}_{L,k,p}^{(u)} < \lambda_{n,T}) \\ &\quad + \sum_{k=1}^{K_L} \Delta \psi_{L,k}(\tau_{j,p}) c_{L,k,p}^{(u)}, \\ &= I + II + III. \end{aligned}$$

By Lemma 2, $I = o_p(1)$, $II = o_p(1)$ as long as $\sqrt{T-1}\lambda_{n,T} \rightarrow 0$, and $III \neq 0$ because $\sum_{k=1}^{K_L} \Delta\psi_{L,k}(t)c_{L,k,p}^{(u)} = \Delta\beta_{\tau_{j,p},p}^{(u)} \neq 0$. The probability of getting $\Delta\tilde{\beta}_{\tau_{j,p},p}^{(u)} = 0$ converges hence to zero.

If $\tau_{j,p} \in \underline{\mathcal{J}}_p$, then

$$|\Delta\tilde{\beta}_{\tau_{j,p},p}^{(u)}|\mathbf{I}_{\overline{\mathcal{J}}_p} + |\Delta\tilde{\beta}_{\tau_{j,p},p}^{(s)}|\mathbf{I}_{\underline{\mathcal{J}}_p} = |\Delta\tilde{\beta}_{\tau_{j,p},p}^{(s)}|.$$

The prove is similar to the case of $\tau_{j,p} \in \overline{\mathcal{J}}_p$ and thus omitted. This completes the proof. \square

Proof of Theorem 3: Recall that the post-Wavelet estimator is obtained by replacing the set of the true jump locations $\tau_{1,1}, \dots, \tau_{S_1+1,1}, \dots, \tau_{1,P}, \dots, \tau_{S_P+1,P}$ in $\hat{\beta}_{(\tau)} = (\hat{\beta}_{\tau_{1,1}}, \dots, \hat{\beta}_{\tau_{S_1+1,1}}, \dots, \hat{\beta}_{\tau_{1,P}}, \dots, \hat{\beta}_{\tau_{S_P+1,P}})'$ by the estimated jump locations $\tilde{\tau} := \{\tilde{\tau}_{j,p} | j \in \{1, \dots, S_p+1\}, p \in \{1, \dots, P\}\}$, given $\tilde{S}_1 = S_1, \dots, \tilde{S}_p = S_p$. By using Theorem 2, we can infer that, conditional on $\tilde{S}_1 = S_1, \dots, \tilde{S}_p = S_p$,

$$\sqrt{n}\mathcal{T}_{(\tilde{\tau})}^{\frac{1}{2}}\hat{\beta}_{(\tilde{\tau})} = \sqrt{n}\mathcal{T}_{(\tau)}^{\frac{1}{2}}\hat{\beta}_{(\tau)} + o_p(1).$$

To study the asymptotic distribution of $\sqrt{n}\mathcal{T}_{(\tilde{\tau})}^{\frac{1}{2}}\hat{\beta}_{(\tilde{\tau})}$ it is hence sufficient to study $\sqrt{n}\mathcal{T}_{(\tau)}^{\frac{1}{2}}\hat{\beta}_{(\tau)}$.

$$\begin{aligned} \hat{\beta}_{(\tau)} &= \left(\sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} \Delta \dot{X}'_{it,(\tau)} \right)^{-1} \left(\sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} \Delta \dot{Y}_{it} \right) \\ &= \beta_{(\tau)} + \left(\sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} \Delta \dot{X}'_{it,(\tau)} \right)^{-1} \left(\sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} \Delta \dot{\epsilon}_{it} \right). \end{aligned}$$

Scaling by $\sqrt{n}\mathcal{T}_{(\tilde{\tau})}^{\frac{1}{2}}$ and rearranging, we get

$$\sqrt{n}\mathcal{T}_{(\tau)}^{\frac{1}{2}}(\hat{\beta}_{(\tilde{\tau})} - \beta_{(\tau)}) = \left((n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} \Delta \dot{X}'_{it,(\tau)} \right)^{-1} \left((n\mathcal{T}_{(\tau)})^{-\frac{1}{2}} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} \Delta \dot{\epsilon}_{it} \right).$$

By Assumption E, the first term on the right hand side converges in probability to $Q_{(\tau)}^\circ$ and the second term converges in distribution to $N(0, V_{(\tau)}^\circ)$. Slutsky's rule implies

$$\sqrt{n}\mathcal{T}_{(\tau)}^{\frac{1}{2}}(\hat{\beta}_{(\tau)} - \beta_{(\tau)}) \xrightarrow{d} N(0, (Q_{(\tau)}^\circ)^{-1}(V_{(\tau)}^\circ)(Q_{(\tau)}^\circ)^{-1}).$$

It follows

$$\sqrt{n}\mathcal{T}_{(\tilde{\tau})}^{\frac{1}{2}}(\hat{\beta}_{(\tilde{\tau})} - \beta_{(\tau)}) = \sqrt{n}\mathcal{T}_{(\tau)}^{\frac{1}{2}}(\hat{\beta}_{(\tau)} - \beta_{(\tau)}) + o_p(1) \xrightarrow{d} N(0, (Q_{(\tau)}^\circ)^{-1}(V_{(\tau)}^\circ)(Q_{(\tau)}^\circ)^{-1}).$$

This completes the Proof. \square

Proof of Proposition 2 Consider $c = 1$ (the case of homoscedasticity without presence of auto- and cross-section correlation). Because by Assumption E, we know that

$$(n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} \Delta \dot{X}'_{it,(\tau)} \xrightarrow{p} Q_{(\tau)}^\circ \quad \text{and}$$

$$(n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T \sum_{j=1}^n \sum_{s=2}^T Z_{it,(\tau)} Z'_{js,(\tau)} \sigma_{ij,ts} \xrightarrow{p} V_{(\tau)}^\circ,$$

it suffices to prove that

$$\hat{V}_{(\bar{\tau})}^{(1)} = (n\mathcal{T}_{(\bar{\tau})})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\bar{\tau})} Z_{it,(\bar{\tau})} \hat{\sigma}^2 \xrightarrow{p} V_{(\tau)}^{(1)},$$

where $V_{(\tau)}^{(1)} = (n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} Z'_{it,(\tau)} \sigma^2$, with $\sigma^2 = E_c(\Delta \dot{e}_{it})$.

$$\begin{aligned} \hat{V}_{(\bar{\tau})}^{(1)} - V_{(\tau)}^{(1)} &= \left(\frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \Delta \hat{e}_{it}^2 - \sigma^2 \right) (n\mathcal{T}_{(\bar{\tau})})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\bar{\tau})} Z'_{it,(\bar{\tau})}, \\ &= +\sigma^2 \left((n\mathcal{T}_{(\bar{\tau})})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\bar{\tau})} Z'_{it,(\bar{\tau})} - (n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} Z'_{it,(\tau)} \right), \\ &= a + b. \end{aligned}$$

From Assumption B(ii), we can infer

$$\begin{aligned} \|a\| &\leq \left(\frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \Delta \hat{e}_{it}^2 - \sigma^2 \right) \frac{1}{n} \sum_{i=1}^n \sum_{t=2}^T \|(\mathcal{T}_{(\bar{\tau})})^{-1/2} Z_{it,(\bar{\tau})}\|^2, \\ &= \left(\frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T ((\Delta \hat{e}_{it}^2 - \Delta e_{it}^2) + (\Delta e_{it}^2 - \sigma^2)) \right) \frac{1}{n} \sum_{i=1}^n \sum_{t=2}^T \|(\mathcal{T}_{(\bar{\tau})})^{-1/2} Z_{it,(\bar{\tau})}\|^2, \end{aligned}$$

From

$$\begin{aligned} \Delta \hat{e}_{it} &= \Delta \dot{Y}_{it} - \Delta \dot{X}'_{it,(\bar{\tau})} \hat{\beta}_{(\bar{\tau})}, \\ &= \Delta \dot{e}_{it} + \Delta \dot{X}'_{it,(\bar{\tau})} (\beta_{(\bar{\tau})} - \hat{\beta}_{(\bar{\tau})}), \end{aligned} \tag{68}$$

and by using Theorem 3 together with Assumption B(ii), we can show that

$$\frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \Delta \hat{e}_{it} - \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \Delta \dot{e}_{it} = o_p(1). \tag{69}$$

By the law of large numbers,

$$\frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=2}^T \Delta \dot{e}_{it} - \sigma^2 = o_p(1).$$

Thus, $\|a\| = (o_p(1) + o_p(1))O_p(1) = o_p(1)$. Moreover, from Theorem 2, we can infer that, given $\tilde{S}_1 = S_1, \dots, \tilde{S}_P = S_P$,

$$(n\mathcal{T}_{(\tilde{\tau})})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tilde{\tau})} Z_{it,(\tilde{\tau})} = (n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} Z_{it,(\tau)} + o_p(1).$$

Thus,

$$\hat{V}_{(\tilde{\tau})}^{(1)} - V_{(\tau)}^{(1)} = o_p(1).$$

Consider $c = 2$ (the case of cross-section heteroskedasticity without auto- and cross-section correlations). Because of Assumption E, it suffices to prove that

$$\hat{V}_{(\tilde{\tau})}^{(2)} = (n\mathcal{T}_{(\tilde{\tau})})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tilde{\tau})} Z_{it,(\tilde{\tau})} \hat{\sigma}_i^2 \xrightarrow{p} V_{(\tau)}^{(2)},$$

where $V_{(\tau)}^{(2)} = (n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} Z_{it,(\tau)} \sigma_i^2$, with $\sigma_i^2 = E_c(\Delta \dot{e}_{it})$.

$$\begin{aligned} \hat{V}_{(\tilde{\tau})}^{(2)} - V_{(\tau)}^{(2)} &= \frac{1}{n} \sum_{i=1}^n (\hat{\sigma}_i^2 - \sigma_i^2) (\mathcal{T}_{(\tilde{\tau})})^{-1} \sum_{t=2}^T Z_{it,(\tilde{\tau})} Z_{it,(\tilde{\tau})}', \\ &= + \frac{1}{n} \sum_{i=1}^n \sigma_i^2 \left((\mathcal{T}_{(\tilde{\tau})})^{-1} \sum_{t=2}^T Z_{it,(\tilde{\tau})} Z_{it,(\tilde{\tau})}' - (\mathcal{T}_{(\tau)})^{-1} \sum_{t=2}^T Z_{it,(\tau)} Z_{it,(\tau)}' \right), \\ &= d + e. \end{aligned}$$

$$\begin{aligned} \|d\| &\leq \frac{1}{n} \sum_{i=1}^n (\hat{\sigma}_i^2 - \sigma_i^2) \sum_{t=2}^T \|(\mathcal{T}_{(\tilde{\tau})})^{-1/2} Z_{it,(\tilde{\tau})}\|^2, \\ &= \frac{1}{n} \sum_{i=1}^n \left((\hat{\sigma}_i^2 - \frac{1}{(T-1)} \sum_{t=2}^T \Delta \dot{e}_{it}) - (\sigma_i^2 - \frac{1}{(T-1)} \sum_{t=2}^T \Delta \dot{e}_{it}) \right) \frac{1}{n} \sum_{t=2}^T \|(\mathcal{T}_{(\tilde{\tau})})^{-1/2} Z_{it,(\tilde{\tau})}\|^2. \end{aligned}$$

From Equation (68), and Theorem 3, we can infer

$$\frac{1}{(T-1)} \sum_{t=2}^T \Delta \hat{e}_{it} - \frac{1}{(T-1)} \sum_{t=2}^T \Delta \dot{e}_{it} = o_p(1) \nu_i, \quad (70)$$

where $\frac{1}{n} \sum_{i=1}^n |\nu_i| = O_p(1)$. Moreover,

$$\sigma_i^2 - \frac{1}{(T-1)} \sum_{t=2}^T \Delta \dot{e}_{it} = o_p(1) \mu_i, \quad (71)$$

where $\frac{1}{n} \sum_{i=1}^n |\mu_i| = O_p(1)$. Note that the first terms in (70) and (71) do not depend on i . By using Assumption B(ii), we can infer

$$\begin{aligned} \|d\| &\leq o_p(1) \frac{1}{n} \sum_{i=1}^n |\nu_i| \sum_{t=2}^T \|(\mathcal{T}_{(\bar{\tau})})^{-1/2} Z_{it,(\bar{\tau})}\|^2 + o_p(1) \frac{1}{n} \sum_{i=1}^n |\mu_i| \sum_{t=2}^T \|(\mathcal{T}_{(\bar{\tau})})^{-1/2} Z_{it,(\bar{\tau})}\|^2, \\ &= o_p(1) O_p(1) + o_p(1) O_p(1). \end{aligned}$$

The proof of e being $o_p(1)$ is similar to the proof of b in the first part. This is because σ_i^2 does not affect the analysis.

The proof of $\hat{V}_{\bar{\tau}}^{(3)}$ being $(n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} Z'_{it,(\tau)} \sigma_{it}^2 + o_p(1)$, with $\sigma_{it}^2 = E_c(\Delta \dot{e}_{it})$ is conceptually similar and thus omitted.

Finally, consider $c = 4$ (The case of cross-section and time heteroskedasticity without auto- and cross-section correlations). As in the previous cases, all we need is to prove that

$$\hat{V}_{(\bar{\tau})}^{(4)} = (n\mathcal{T}_{(\bar{\tau})})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\bar{\tau})} Z'_{it,(\bar{\tau})} \Delta \hat{e}_{it}^2 \xrightarrow{p} V_{(\tau)}^{(4)},$$

where

$$V_{(\tau)}^{(4)} = (n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} Z'_{it,(\tau)} \sigma_{it}^2,$$

with $\sigma_{it}^2 = E_c(\Delta \dot{e}_{it})$.

$$\begin{aligned} \hat{V}_{(\bar{\tau})}^{(4)} - V_{(\tau)}^{(4)} &= (n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\bar{\tau})} Z'_{it,(\bar{\tau})} (\Delta \hat{e}_{it}^2 - \Delta \dot{e}_{it}^2) \\ &\quad + (n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T (Z_{it,(\bar{\tau})} Z'_{it,(\bar{\tau})} - Z_{it,(\tau)} Z'_{it,(\tau)}) \Delta \dot{e}_{it}^2 \\ &\quad + (n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T Z_{it,(\tau)} Z'_{it,(\tau)} (\Delta \dot{e}_{it}^2 - \sigma_{it}^2). \\ &= f + g + h. \end{aligned}$$

Cauchy-Schwarz inequality implies

$$\|f\| \leq \left((n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T \|Z_{it,(\tau)}\|^2 \right)^{1/2} \left((n\mathcal{T}_{(\tau)})^{-1} \sum_{i=1}^n \sum_{t=2}^T (\Delta \hat{e}_{it}^2 - \Delta \dot{e}_{it}^2) \right)^{1/2} = o_p(1).$$

By using Theorem 3, we can also verify that $\|g\| = o_p(1)$. Finally, Cauchy-Schwarz, Assumption B(ii), the law of large numbers implies that $\|h\| = o_p(1)$. It follows

$$\hat{V}_{(\bar{\tau})}^{(4)} \xrightarrow{p} V_{(\tau)}^{(4)}.$$

This completes the proof. \square

B Supplementary Material: Inference on the Detected Jumps

To test for the statistical significance of the post-SAW detected break points, we propose two testing procedures: A Chow-type test to individually examine the significance of the jumps and a Hotelling-type test to examine whether or not a model with constant parameters is more appropriate for the data than a model with the detected jumps.

B.1 Chow-type Test

Based on the asymptotic distribution of $\hat{\beta}_{(\tilde{\tau})}$ derived in Theorem 3 and the consistent variance estimators $\hat{\Sigma}_{(\tilde{\tau})}^{(c)}$ presented in Proposition 2, we can construct a Chow-type test to examine the statistical significance of the detected jumps. To simplify exposition, let us begin with a simple case of one regressor and a single jump that is detected by our post-SAW approach at some $\tilde{\tau}_1 \in \{3, \dots, T-1\}$. The post-SAW regression model can be written as follows

$$\Delta \dot{Y}_{it} = \Delta \dot{X}_{it}^{(1)} \beta_{\tilde{\tau}_1} + \Delta \dot{X}_{it}^{(2)} \beta_{\tilde{\tau}_2} + \Delta \dot{e}_{it}, \quad (72)$$

where

$$\Delta \dot{X}_{it}^{(j)} = \Delta \dot{X}_{it} \mathbf{I}(\tilde{\tau}_{j-1} < t \leq \tilde{\tau}_j)$$

with $\tilde{\tau}_0 = 2$ and $\tilde{\tau}_2 = T$, for $j = 1, 2$.

In vector notation, Model (72) can be written as

$$\begin{pmatrix} \Delta \dot{Y}_{i2} \\ \vdots \\ \Delta \dot{Y}_{i\tilde{\tau}_1} \\ \Delta \dot{Y}_{i,\tilde{\tau}_1+1} \\ \vdots \\ \Delta \dot{Y}_{iT} \end{pmatrix} = \begin{pmatrix} \Delta \dot{X}_{i2} & 0 \\ \vdots & \vdots \\ \Delta \dot{X}_{i\tilde{\tau}_1} & 0 \\ 0 & \Delta \dot{X}_{i,\tilde{\tau}_1+1} \\ \vdots & \vdots \\ 0 & \Delta \dot{X}_{iT} \end{pmatrix} (\beta_{\tilde{\tau}_1}, \beta_{\tilde{\tau}_2}) + \begin{pmatrix} \Delta \dot{e}_{i2} \\ \vdots \\ \Delta \dot{e}_{i\tilde{\tau}_1} \\ \Delta \dot{e}_{i,\tilde{\tau}_1+1} \\ \vdots \\ \Delta \dot{e}_{iT} \end{pmatrix}.$$

Let

$$\beta_{\tilde{\tau}_2}^* = \beta_{\tilde{\tau}_2} - \beta_{\tilde{\tau}_1}.$$

Because

$$\begin{pmatrix} \Delta \dot{X}_{i1} & 0 \\ \vdots & \vdots \\ \Delta \dot{X}_{i\tilde{\tau}_1} & 0 \\ 0 & \Delta \dot{X}_{i,\tilde{\tau}_1+1} \\ \vdots & \vdots \\ 0 & \Delta \dot{X}_{i,\tilde{\tau}_2} \end{pmatrix} (\beta_{\tilde{\tau}_1}, \beta_{\tilde{\tau}_2}) = \begin{pmatrix} \Delta \dot{X}_{i1} & 0 \\ \vdots & \vdots \\ \Delta \dot{X}_{i\tilde{\tau}_1} & 0 \\ \Delta \dot{X}_{i,\tilde{\tau}_1+1} & \Delta \dot{X}_{i,\tilde{\tau}_1+1} \\ \vdots & \vdots \\ \Delta \dot{X}_{i,\tilde{\tau}_2} & \Delta \dot{X}_{i,\tilde{\tau}_2} \end{pmatrix} (\beta_{\tilde{\tau}_1}, \beta_{\tilde{\tau}_2}^*),$$

we can re-write (72) as

$$\Delta \dot{Y}_{it} = \Delta \dot{X}_{it} \beta_{\tilde{\tau}_1} + \Delta \dot{X}_{it}^{(2)} \beta_{\tilde{\tau}_2}^* + \Delta \dot{e}_{it}. \quad (73)$$

If $\tilde{\tau}_1$ is statistically insignificant, then $\beta_{\tilde{\tau}_2}^*$ theoretically should be zero. The test problem can hence be formulated as follows:

$$\begin{aligned} H_0 : \quad \beta_{\tilde{\tau}_2}^* &= \beta_{\tilde{\tau}_2} - \beta_{\tilde{\tau}_1} = 0 \\ H_1 : \quad \beta_{\tilde{\tau}_2}^* &\neq 0, \end{aligned}$$

and the test statistic can be derived from the asymptotic distribution of $\hat{\beta}_{\tilde{\tau}_2}^* = \hat{\beta}_{\tilde{\tau}_2} - \hat{\beta}_{\tilde{\tau}_1}$, where $\hat{\beta}_{\tilde{\tau}_2}$ and $\hat{\beta}_{\tilde{\tau}_1}$ are the post-SAW estimators of $\beta_{\tilde{\tau}_2}$ and $\beta_{\tilde{\tau}_1}$ respectively.

Now that we have motivated the testing idea for a straightforward case with an unique explanatory variable and a single detected jump, we turn to generalizations. The testing procedure is similar to the above discussed procedure and can be based on an analogous re-transformation of the multivariate post-wavelet Model (38):

$$\Delta \dot{Y}_{it} = \Delta \sum_{p=1}^P \dot{X}_{it,p} \beta_{\tilde{\tau}_{1,p}} + \sum_{p=1}^P \sum_{j=2}^{\tilde{S}_p+1} \Delta \dot{X}_{it,p}^{*(j)} \beta_{\tilde{\tau}_{j,p}}^* + \Delta \dot{e}_{it}, \quad (74)$$

where

$$\Delta \dot{X}_{it,p}^{*(j)} = \Delta \dot{X}_{it,p} \mathbf{I}(t > \tilde{\tau}_{j-1,p}),$$

and

$$\beta_{\tilde{\tau}_{j,p}}^* = \beta_{\tilde{\tau}_{j,p}} - \beta_{\tilde{\tau}_{j-1,p}}.$$

The test statistic takes the form

$$CH_{\tilde{\tau}_{j,p}} = \sqrt{n \hat{\Theta}_{\tilde{\tau}_{j,p}}^{-1}} (\hat{\beta}_{\tilde{\tau}_{j,p}} - \hat{\beta}_{\tilde{\tau}_{j-1,p}}) \xrightarrow{d} N(0, 1), \quad (75)$$

where

$$\hat{\Theta}_{\tilde{\tau}_{j,p}} = \frac{\hat{\Sigma}_{\tilde{\tau}_{j,p}, \tilde{\tau}_{j,p}}}{(\tilde{\tau}_{j+1,p} - \tilde{\tau}_{j,p})} + \frac{\hat{\Sigma}_{\tilde{\tau}_{j-1,p}, \tilde{\tau}_{j-1,p}}}{(\tilde{\tau}_{j,p} - \tilde{\tau}_{j-1,p})} - 2 \frac{\hat{\Sigma}_{\tilde{\tau}_{j,p}, \tilde{\tau}_{j-1,p}}}{(\tilde{\tau}_{j+1,p} - \tilde{\tau}_{j,p})^{1/2} (\tilde{\tau}_{j,p} - \tilde{\tau}_{j-1,p})^{1/2}}$$

with $\hat{\Sigma}_{\tau_{j,p}, \tau_{j,p}}$, $\hat{\Sigma}_{\tau_{j-1,p}, \tau_{j-1,p}}$, and $\hat{\Sigma}_{\tau_{j,p}, \tau_{j-1,p}}$ are the elements of $\hat{\Sigma}_{(\tilde{\tau})}$ corresponding to the variances and correlation estimates of $\beta_{\tau_{j,p}}$ and $\beta_{\tau_{j-1,p}}$ respectively.

We reject the null hypothesis at a level α if $|CH_{\tilde{\tau}_{j,p}}|$ is larger than the $(1 - \alpha/2)$ -quantile of the standard normal distribution.

B.2 Hotelling-type Test

To test whether a model with constant parameters is more appropriate for the data than a model with the post-SAW detected jumps, we examine the following multidimensional testing problem:

$$\begin{aligned} H_0 : & \beta_{\tilde{\tau}_{1,1}}^* = \dots = \beta_{\tilde{\tau}_{S_P,P}}^* = 0 \\ H_1 : & \beta_{\tilde{\tau}_{j,p}}^* \neq 0 \text{ for at least on } \tilde{\tau}_{j,p}. \end{aligned}$$

In this case, the test can take the form of a Hotelling test statistic with an asymptotic χ^2 distribution:

$$\mathbb{T}_{(\tilde{\tau})}^2 = n(\tilde{\mathbb{D}}\hat{\beta}_{(\tilde{\tau})})' (\tilde{\mathbb{D}}\mathcal{T}_{(\tilde{\tau})}^{-\frac{1}{2}}\hat{\Sigma}_{(\tilde{\tau})}\mathcal{T}_{(\tilde{\tau})}^{-\frac{1}{2}}\mathbb{D}_{(\tilde{\tau})}')^{-1}(\tilde{\mathbb{D}}\hat{\beta}_{(\tilde{\tau})}) \xrightarrow{d} \chi^2\left(\sum_{p=1}^P \tilde{S}_p\right),$$

where

$$\tilde{\mathbb{D}} = \begin{pmatrix} \mathbf{D}_{\tilde{S}_1 \times (\tilde{S}_1+1)} & 0_{\tilde{S}_1 \times (\tilde{S}_2+1)} & \cdots & 0_{\tilde{S}_1 \times (\tilde{S}_P+1)} \\ 0_{\tilde{S}_2 \times (\tilde{S}_1+1)} & \mathbf{D}_{\tilde{S}_2 \times (\tilde{S}_2+1)} & & 0_{\tilde{S}_2 \times (\tilde{S}_P+1)} \\ \vdots & & \ddots & \vdots \\ 0_{\tilde{S}_P \times (\tilde{S}_1+1)} & 0_{\tilde{S}_P \times (\tilde{S}_2+1)} & \cdots & \mathbf{D}_{\tilde{S}_P \times (\tilde{S}_P+1)} \end{pmatrix}.$$

Here, $0_{\tilde{S}_p \times (\tilde{S}_k+1)}$ are $(\tilde{S}_p \times (\tilde{S}_k+1))$ -matrices with zero elements, and $\mathbf{D}_{\tilde{S}_p \times (\tilde{S}_p+1)}$ are $(\tilde{S}_p \times (\tilde{S}_p+1))$ -matrices defined as follows:

$$\mathbf{D}_{\tilde{S}_p \times (\tilde{S}_p+1)} = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & & & \\ 0 & 0 & \cdots & \cdots & & -1 & 1 \end{pmatrix}.$$

We reject the null hypothesis at a level α if $\mathbb{T}_{(\tilde{\tau})}^2$ is larger than the $(1 - \alpha)$ -quantile of the $\chi^2(\sum_{p=1}^P \tilde{S}_p)$ - distribution.

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