# Semiparametric Estimations under Shape Constraints 

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#### Abstract

Economic theory provides the econometrician with substantial structure and restrictions necessary to give economic interpretation to empirical findings. In many settings, such as those in consumer demand and production studies, these restrictions often take the form of monotonicity and curvature constraints. Although such restrictions may be imposed in certain parametric empirical settings in a relatively straightforward fashion by utilizing parametric restrictions or particular parametric functional forms (Cobb-Douglas, CES, etc.), imposing such restrictions in semiparametric models is often problematic. Our paper provides one solution to this problem by incorporating penalized splines, where curvature constraints are maintained via integral transformations of spline basis expansions. We derive the estimator, algorithms for its solution, and its large sample properties. Inferential procedures are discussed as well as methods for selecting the smoothing parameter. We also consider multiple regressions under the framework of additive models. We conduct a series of Monte Carlo simulations to illustrate the finite sample properties of the estimator. We apply the proposed methods to estimate two canonical relationships, one in consumer behavior and one in producer behavior. These two empirical settings examine the relationship between individuals' degree of optimism and risk tolerance and a production function with multiple inputs.


Keywords: monotonicity, shape constraints, semiparametric econometrics, smoothing splines, willingness to pay, production functions
JEL Classification numbers: C14, C15, D4, D6

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## 1 Introduction

Economic theories can provide useful guidance on the modeling of real world data. Utility functions associated with rational preference are monotone; under convex preferences utility function a quasiconcave. Demand functions of normal goods are downward sloping (Matzkin, 1991; Lewbel, 2010; Blundell et al., 2012). According to duality theory, profit functions are concave in output price and cost functions are monotonically increasing and concave in input price. Convex function estimation is also used extensively in derivative asset pricing models (Broadie et al, 2000; AïtSahalia and Duarte, 2003; Yatchew and Härdle, 2006). Researchers, when trying to model economic relationships, often face at least two challenges. One is fidelity to economic theory. Another is flexibility in functional forms (Guilkey, Lovell, and Sickles, 1983, Diewert and Wales, 1987). In addition, these two goals are often at odds: conformity to theories often dictates relatively rigid functional forms, while flexible parameterizations sometimes lead to implausible predictions.

One fruitful approach to tackle this dilemma is to use nonparametric or semiparametric methods subject to the restrictions suggested by economic theory. This is a well-developed literature and has had a number of contributors. Matzkin (1994) and Yatchew (2003, Chapter 6) provide general reviews of this literature. For recent developments, see Hall and Huang (2001), Groeneboom et al. (2001), Mammen and Horowitz (2004), Carroll et al. (2011), Shively et al. (2011), and Blundell et al. (2012), among others. We follow in this line of research and present a flexible semiparametric estimator with shape constraints. We focus on functional relationships with two shape constraints: monotonicity and concavity (convexity) as this is the class of functions encountered most frequently in economic studies. Functional relationships with either one of these two constraints are special cases of our estimator.

We base our work on Ramsay's (1998) monotone smooth estimator and utilize integral transformations defined by a certain set of differential equations to impose shape restrictions. A key advantage of this transformation approach is that it transforms a constrained problem into an unconstrained one. We subsequently model the unconstrained problem using penalized spline methods, resulting in a nonlinear semiparametric estimator. We show that careful choice of transformation and of the model-based penalty can simplify estimation considerably.

We propose an iterative algorithm to calculate the proposed estimator. We establish the consistency of the estimator and present approximate methods for inference and for selecting the smoothing parameter. We then extend our estimator to an additive model. We illustrate the finite sample performance and usefulness of our methods with Monte Carlo simulations and two empirical applications.

The remainder of the paper is organized as follows. Section 2 briefly reviews the relevant literature and then presents our transformation-based model to accommodate shape restrictions. Section 3 proposes a Gauss-Jordan algorithm to solve the estimator. Sections 4 and 5 discuss methods of inferences and model specification. Section 6 extends the model to multiple regressions. Sections 7 and 8 report Monte Carlo simulations and two empirical examples. The last section concludes. A technical appendix gathers all proofs.

## 2 Model and Estimator

Several approaches have been used to impose restrictions in statistical estimations. A simple approach is the transformation of variables. For instance, the logarithmic transformation is commonly used to assure positiveness of predicted outcomes and the Box-Cox transformation can offer an even more flexible alternative. In the estimation of production functions, the Cobb-Douglas, constant elasticity of substitution (CES), translog, and generalized Leontief specifications are commonly employed. These functional forms are often chosen because they satisfy certain theoretical properties and also out of their simplicity, as they are either linear in parameters after a simple log transformation or are linear to begin with. Simple parametric forms, however, can sometimes entail nontrivial restrictions. For example, a logarithm transformation of the dependent variable implies multiplicative errors rather than the usual additive ones.

To avoid rigid functional forms, semiparametric and nonparametric methods have been used to accommodate shape restrictions. An early example is Brunk's (1955) isotonic estimator, which essentially produces a monotone step function. Mukerjee (1988) and Mammen (1991) developed kernel-based isotonic regression techniques which consist of a kernel smoothing step and an isotonization step to maintain monotonicity. Instead of isotonization, Hall and Huang (2001) suggested a penalized kernel method to obtain monotonicity. Their
method is employed by Henderson et al. (2012), Blundell et al. (2012), Ma and Racine (2013) and Du et al. (forthcoming) for various applications or further generalizations. Another popular family of smoothers, the spline-based methods, has also been used by Ramsay (1988), Kelly and Rice (1990), and Mammen and Thomas-Agnam (1999) who proposed monotone estimators based on shape preserving spline basis functions. A third possibility is to use the technique of rearrangement or data sharpening (cf. Braun and Hall (2001) and Chernozhukov, et al. (2009)). Shively et al. (2009) consider a Bayesian approach of nonparametric monotone function estimation of Gaussian regressions, which is generalized to log-concave likelihood functions by Shively et al. (2011).

Our estimator is inspired by the smooth monotone estimator of Ramsay (1998). Suppose $y=f(x)$ is a smooth monotone function of $x$. For simplicity, we assume that $x \in[0,1]$. Ramsay (1998) proposed to model an unknown monotone function via the following integral transformation:

$$
\begin{equation*}
f(x)=\int_{0}^{x} \exp (r(s)) d s \tag{1}
\end{equation*}
$$

where $r$ is a square integrable function on $[0,1]$. Since $f^{\prime}(x)=\exp (r(x))>0$ for all $x$, the monotone restriction is satisfied. Unlike some penalty-based monotone estimators that impose observation-specific monotonicity, (1) is globally monotone thanks to the positive exponential functional embedded in the integral transformation.

Since $f^{\prime \prime}(x)=f^{\prime}(x) r^{\prime}(x)$ and $f^{\prime}(x)>0, f(x)$ is concave if $r^{\prime}(x)<0$ for all $x$. Our strategy is to use an additional integration transformation (1) to impose the condition that $r^{\prime}(x)<0$. In particular, we consider the following parameterization

$$
\begin{equation*}
f(x)=\int_{0}^{x} \exp \left(-\int_{0}^{s} g(t) d t\right) d s \tag{2}
\end{equation*}
$$

It follows that $f^{\prime}(x)=\exp \left(-\int_{0}^{x} g(t) d t\right)>0$ and $f^{\prime \prime}(x)=-f^{\prime}(x) g(x)$, implying that $f^{\prime \prime}(\cdot)<0$ if $g(\cdot)>0$. Thus under (2), the monotonicity and concavity constraints are reduced to a simple positiveness constraint that $g(x)>0$ for all $x$. Natural candidates of $g$ include $g(x)=x^{2}$ and $g(x)=\exp (x)$; other choices are certainly possible. Below we will show that $g(x)=x^{2}$ is particularly appealing for the proposed method on theoretical and practical grounds.

The parameterization (2) can be characterized by the following differential equation

$$
g(x)=-\frac{f^{\prime \prime}(x)}{f^{\prime}(x)} \cdot{ }^{1}
$$

The solution is given by

$$
f(x)=\beta_{0}+\beta_{1} \int_{0}^{x} \exp \left(-\int_{0}^{s} g(t) d t\right) d s
$$

where $\beta_{0}$ and $\beta_{1}$ are generic constants.
Given an iid random sample $\left\{Y_{i}, X_{i}\right\}_{i=1}^{n}$ with $X_{i} \in[0,1]$, we can consider the following statistical model for a monotone and concave functional relationship

$$
\begin{equation*}
Y_{i}=f\left(X_{i}\right)+e_{i}=\beta_{0}+\beta_{1} \int_{0}^{X_{i}} \exp \left(-\int_{0}^{s} g(t) d t\right) d s+e_{i} \tag{3}
\end{equation*}
$$

where $g(\cdot)>0$ and $e_{i}$ are iid error terms with mean zero and a finite variance $\sigma^{2}$. We will model $g(\cdot)$ by $g \circ h(\cdot)$, where $h$ is a square integrable function defined on $[0,1]$ free of constraints.

One major advantage of the transformation-based approach to incorporate constraints is that we can transform a constrained problem into an unconstrained one. In our case, this reduces to the modeling of $h$. Lacking theoretical guidance or a priori information on $h$, we opt to model $h$ using a flexible nonparametric estimator. Specifically, we use the spline method, in which it is relatively straightforward to embed smoothers for nonlinear functionals or to implement additive structures in multiple regressions using splines. Since the spline is a piecewise polynomial that is smoothly connected at its joints (knots), then because of their local nature splines do not suffer from the oscillations associated with global polynomials such as the power series.

There exist many types of splines, such as the truncated power series, $B$-splines, radial splines, periodic splines and thin-plate splines (cf. de Boor (2001) for a general treatment of splines). Let $0<k_{1}<\cdots<k_{M}<1$ be a series of knots of the spline basis functions. The

[^1]popular truncated power series splines are given by
$$
\Phi(x)=\left(1, x, \ldots, x^{p},\left(x-k_{1}\right)_{+}^{p}, \cdots,\left(x-k_{M}\right)_{+}^{p}\right)^{T},
$$
where $(x)_{+}=\max (x, 0)$, and $p$ is a positive integer. Define $h(x)=c^{T} \Phi(x)$ with $c$ being a vector of coefficients with compatible dimension. This construction, a linear combination of spline basis functions, is a flexible tool of curve fitting. The degree of smoothness of spline approximation is controlled by $p$ : a linear combination of spline basis functions of degree $p$ is a $p$ th degree polynomial on each subinterval $\left[k_{m}, k_{m+1}\right]$ and has $p-1$ continuous derivatives on its entire domain. The global polynomials control the overall shape of a curve, while the spline basis functions reflect local features. For flexibility and numerical stability, a common practice in spline approximation is to employ a large number of low order spline basis functions (i.e., large $M$, small $p$ ).

In practice, truncated power series are often transformed to $B$-splines, which are the maximally differentiable interpolative basis functions. The $B$-splines are generalizations of Bézier curve and can be constructed recursively (cf. Eilers and Marx (1996)). B-splines sometimes facilitate theoretical analysis and usually produce better finite sample performance.

Let $P=1+p+M$ and $\Phi$ be a $P$-dimensional basis functions. We consider the following model

$$
\begin{equation*}
Y_{i}=f\left(X_{i} ; \beta, c\right)+e_{i}=\beta_{0}+\beta_{1} \int_{0}^{X_{i}} \exp \left(-\int_{0}^{s} g\left(c^{T} \Phi(t)\right) d t\right) d s+e_{i} \tag{4}
\end{equation*}
$$

The intercept $\beta_{0}$ and a slope-type parameter $\beta_{1}$ are required for identification as the parameterization of $f$ does not allow for free location and scale parameters. To see this, consider the simplest case $g(x)=a$, where $a$ is a non-zero constant. It follows that $f(x)=(1-\exp (-a x)) / a$, whose location and scale can not independently vary.

Model (4) is a semiparametric model with two parametric coefficients and a nonparametric smoother $g$. To balance fidelity to the data and smoothness of the estimator, we adopt the approach of penalized spline estimation. ${ }^{2}$ This method uses a relatively generous spline basis and shrinks all coefficients towards zero to avoid overfitting. We choose this approach because the delicate balance between goodness-of-fit and smoothness is governed by a single

[^2]smoothing parameter and therefore easier to implement. ${ }^{3}$
To implement this estimator for model (4) we use penalized least squares, minimizing the sum of squared residuals plus a penalty on the roughness of $f$. The objective function is given by
\[

$$
\begin{equation*}
Q_{\lambda}(\beta, c)=\frac{1}{n} \sum_{i=1}^{n}\left(Y_{i}-f\left(X_{i} ; \beta, c\right)\right)^{2}+\lambda D(f) \tag{5}
\end{equation*}
$$

\]

where $D(f)>0$ reflects the roughness of $f$. For the $p$ th degree splines, a popular choice of the penalty is the integrated squared $q$ th derivative of $f, q \leq p$. For example, the integrated quadratic penalty with $q=2$ is commonly used, which leads to the natural cubic splines in smoothing splines.

In penalized spline estimations, we can in principle select the basis functions and the penalty separately. Nonetheless, for nonlinear models, careful choice of penalty with respect to the form of $f$ can sometimes improve the estimation considerably. For instance, Heckman and Ramsay (2000) showed that proper model-based penalties can reduce the number of spline basis functions and the approximation bias at the same time, resulting in smaller mean square errors. In our case a natural choice of the penalty is the integrated relative curvature; that is, $D(f)=-\int_{0}^{1} f^{\prime \prime}(x) / f^{\prime}(x) d x=\int_{0}^{1} g(x) d x$. Since $g(x)>0$ by construction, it consists a valid roughness penalty. This penalty on the relative curvature penalizes not only the curvature of $f$ but also small values of $f^{\prime}$. Consequently, it prevents the 'boundary' solutions where $f^{\prime}(x)=0$.

## 3 Estimation Algorithm

Denote the solution to the proposed nonlinear estimation (5) by $\hat{\beta}=\left(\hat{\beta}_{0}, \hat{\beta}_{1}\right)^{T}$ and $\hat{c}$. Let $m(x ; c)=\int_{0}^{x} \exp \left(-\int_{0}^{s} g\left(c^{T} \Phi(t)\right) d t\right) d s$. It follows that $D(f)=D(m)$. Define $\hat{m}\left(X_{i}\right)=$ $m\left(X_{i} ; \hat{c}\right)$ and $g^{\prime}(x)=d g(x) / d x$. Replacing $\beta$ with $\hat{\beta}$ and applying Taylor expansion to $m$ in

[^3](5) with respect to $c$ around $\hat{c}$ yields
\[

$$
\begin{equation*}
\frac{1}{n} \sum_{i=1}^{n}\left(Y_{i}-\hat{\beta}_{0}-\hat{\beta}_{1} m\left(X_{i} ; \hat{c}\right)-\hat{\beta}_{1} \hat{Z}_{i}(c-\hat{c})\right)^{2}+\lambda D \tag{6}
\end{equation*}
$$

\]

where

$$
\hat{Z}_{i}=\frac{\partial \hat{m}\left(X_{i} ; \hat{c}\right)}{\partial c}=-\int_{0}^{X_{i}}\left\{\int_{0}^{s}\left(\Phi(t) g^{\prime}\left(\hat{c}^{T} \Phi(t)\right) d t\right) \exp \left(\int_{0}^{s}-g\left(\hat{c}^{T} \Phi(t)\right) d t\right)\right\} d s
$$

The first order condition of (6) with respect to $c$ is given by

$$
\begin{equation*}
-\frac{1}{n} \sum_{i=1}^{n} \hat{\beta}_{1} \hat{Z}_{i}^{T}\left(Y_{i}-\hat{\beta}_{0}-\hat{\beta}_{1} \hat{m}\left(X_{i}\right)-\hat{\beta}_{1} \hat{Z}_{i}(c-\hat{c})\right)+\lambda D^{\prime}=0 \tag{7}
\end{equation*}
$$

where

$$
D^{\prime}=\frac{\partial D}{\partial c}=\int_{0}^{1} \Phi(x) g^{\prime}\left(c^{T} \Phi(x)\right) d x
$$

Next denote $\hat{D}=D(\hat{m})$ and $\hat{D}^{\prime}$ and $\hat{D}^{\prime \prime}$ its first and second derivatives with respect to $c$ evaluated at $\hat{c}$. Taking a Taylor expansion of $D^{\prime}$ with respect to $c$ around $\hat{c}$ yields

$$
\begin{equation*}
-\frac{1}{n} \sum_{i=1}^{n} \hat{\beta}_{1} \hat{Z}_{i}^{T}\left(Y_{i}-\hat{\beta}_{0}-\hat{\beta}_{1} \hat{m}\left(X_{i}\right)-\hat{\beta}_{1} \hat{Z}_{i}(c-\hat{c})\right)+\lambda \hat{D}^{\prime}+\lambda \hat{D}^{\prime \prime}(c-\hat{c}) \approx 0 \tag{8}
\end{equation*}
$$

Define $\hat{e}_{i}=Y_{i}-\hat{\beta}_{0}-\hat{\beta}_{1} \hat{m}\left(X_{i}\right)$. Substituting $\hat{e}_{i}$ into (7) and rearranging terms yield

$$
\begin{equation*}
\left(\frac{1}{n} \sum_{i=1}^{n} \hat{\beta}_{1}^{2} \hat{Z}_{i}^{T} \hat{Z}_{i}+\lambda \hat{D}^{\prime \prime}\right)(c-\hat{c}) \approx \frac{1}{n} \sum_{i=1}^{n} \hat{\beta}_{1} \hat{Z}_{1}^{T} \hat{e}_{i}-\lambda \hat{D}^{\prime} \tag{9}
\end{equation*}
$$

Expression (9) suggests a Gauss-Jordan iterative algorithm to solve for the proposed estimator. Let $\hat{c}_{-}$be the current estimate of $c$ and $\hat{m}\left(X_{i}\right), \hat{Z}_{i}, \hat{D}^{\prime}, \hat{D}^{\prime \prime}$ and $\hat{e}_{i}$ be evaluated at $c=\hat{c}_{-}$. Denote $Y=\left(Y_{1}, \ldots, Y_{n}\right)^{T}$ and $\hat{m}=\left(\hat{m}\left(X_{1}\right), \ldots, \hat{m}\left(X_{n}\right)\right)^{T}$. Taking $\hat{m}$ as given, we calculate $\hat{\beta}$ via the ordinary least squares by regressing $Y$ on $\hat{m}$ and a constant one. Next holding $\hat{\beta}$ constant, we update $c$ according to the following formula:

$$
\begin{equation*}
\hat{c}=\hat{c}_{-}+\left\{\frac{1}{n} \hat{\beta}_{1}^{2} \hat{Z}^{T} \hat{Z}+\lambda \hat{D}^{\prime \prime}\right\}^{-1}\left\{\frac{1}{n} \hat{\beta}_{1} \hat{Z}^{T} \hat{e}-\lambda \hat{D}^{\prime}\right\} \tag{10}
\end{equation*}
$$

where $\hat{e}=\left(\hat{e}_{1}, \ldots, \hat{e}_{n}\right)^{T}$ and $\hat{Z}=\left(\hat{Z}_{1}^{T}, \ldots, \hat{Z}_{n}^{T}\right)^{T} . \hat{\beta}$ and $\hat{c}$ are updated alternatively in this fashion until convergence.

Remark 1. The penalty $D(\hat{m})$ and their derivatives $\hat{D}^{\prime}$ and $\hat{D}^{\prime \prime}$ generally depend on the current estimate $\hat{c}_{-}$and therefore need to be recalculated at each stage of the updating. This updating process is simplified when $g(x)=\frac{1}{2} x^{2}$. Recall that $h(x)=c^{T} \Phi(x)$. Define $K=$ $\int_{0}^{1} \Phi(x) \Phi^{T}(x) d x$. It follows that $D(m)=\frac{1}{2} c^{T} K c$ and the updating formula (8) simplifies to

$$
\hat{c}=\hat{c}_{-}+\left\{\frac{1}{n} \hat{\beta}_{1}^{2} \hat{Z}^{T} \hat{Z}+\lambda K\right\}^{-1}\left\{\frac{1}{n} \hat{\beta}_{1} \hat{Z}^{T} \hat{e}-\lambda K \hat{c}_{-}\right\} .
$$

Thus with a quadratic $g$, the penalty weight matrix remains a constant that does not depend on unknown parameters. Moreover, the Taylor expansion given by (8) is exact.

Remark 2. The convergence of the estimation is usually quite speedy. To assure that each step improves the penalized objective function, we also implement a step-halving procedure. Whenever an updating step in c fails to improve the objective function (6), we divide it by two to mitigate overshooting. This adjustment further improves the numerical stability of the proposed algorithm.

## 4 Large Sample Properties and Inferences

Despite the popularity of penalized spline methods, their theoretical properties are less well understood. Early results were provided in Wand (1999), Aerts et al. (2002) and Yu and Ruppert (2002) under the framework that the dimension of the spline basis is sufficiently large and fixed. Hall and Opsomer (2005) investigated the problem using a white noise representation. Claeskens et al. (2008) showed that if the number of knots increases as sample size increases, then the asymptotic properties of penalized splines share many characteristics of the asymptotic distributions of regression splines and smoothing splines. ${ }^{4}$ Kauermann et al. (2009) studied the asymptotic properties of penalized splines for generalized linear

[^4]models under the regression splines scenario. Li and Ruppert (2008) also used the device of equivalent kernels to study smoothing splines.

Following Wand (1999), Aerts et al. (2002) and Yu and Ruppert (2002), we study the asymptotic behavior of the proposed methods under the premise that the fixed number of spline basis functions is sufficiently large that the approximation error is $o(1)$. As in nonparametric modeling, the model is flexible enough to adapt to regression functions of unknown form; at the same time as in parametric modeling, the number of parameter is fixed, and the parameters can be estimated at $\sqrt{n}$ rates. This type of fixed-knot asymptotics converge to a known normal distribution and thus provide standard inferential benchmarks.

To facilitate the derivation, we first present an alternative representation of solution (10). Given current estimates $\hat{\beta}$ and $\hat{c}_{-}$, define the 'pseudo regressand' $\tilde{Y}_{i}=Y_{i}-\hat{\beta}_{0}-\hat{\beta}_{1} \hat{m}\left(X_{i}\right)+$ $\hat{\beta}_{1} \hat{Z}_{i} \hat{c}_{-}$. Plugging $\tilde{Y}_{i}$ into (7) and rearranging terms yield

$$
\left(\frac{1}{n} \hat{\beta}_{1}^{2} \hat{Z}^{T} \hat{Z}+\lambda \hat{D}^{\prime \prime}\right) \hat{c} \approx \frac{1}{n} \hat{\beta}_{1} \hat{Z}^{T} \tilde{Y}+\lambda\left(\hat{D}^{\prime}-\hat{D}^{\prime \prime} \hat{c}_{-}\right)
$$

where $\tilde{Y}=\left(\tilde{Y}_{1}, \ldots, \tilde{Y}_{n}\right)^{T}$. Holding $\hat{\beta}$ constant, we can update $c$ using the following alternative formula:

$$
\begin{equation*}
\hat{c}=\left(\frac{1}{n} \hat{\beta}_{1}^{2} \hat{Z}^{T} \hat{Z}+\lambda \hat{D}^{\prime \prime}\right)^{-1}\left(\frac{1}{n} \hat{\beta}_{1} \hat{Z}^{T} \tilde{Y}+\lambda\left(\hat{D}^{\prime}-\hat{D}^{\prime \prime} \hat{c}_{-}\right)\right) . \tag{11}
\end{equation*}
$$

Remark 3. When $g=\frac{1}{2} x^{2}$, we have $D(m)=\frac{1}{2} c^{T} K c$ and $D^{\prime}-D^{\prime \prime} c=0$, resulting in a simpler updating process

$$
\hat{c}=\left(\frac{1}{n} \hat{\beta}_{1}^{2} \hat{Z}^{T} \hat{Z}+\lambda K\right)^{-1}\left(\frac{1}{n} \hat{\beta}_{1} \hat{Z}^{T} \tilde{Y}\right) .
$$

Since $\hat{\beta}, \hat{Z}$ and $\tilde{Y}$ all depend on the current estimate $\hat{c}_{-}$, iterations are still called for.
Remark 4. We present the alternative representation (11) to facilitate the asymptotic analysis. Our numerical experiments indicate that the Gauss-Jordan algorithm given in the previous section is usually more robust and converges faster, especially when a non-qudratic $g$ is used. We recommend the Gauss-Jordan algorithm for the calculation of our estimator.

This representation (11) of $c$ as a linear function of $\tilde{Y}$ allows us to use known results on linear smoothers for inferences. Denote $\theta(\lambda)=(\beta(\lambda), c(\lambda))$. We emphasize the dependence
of the estimator on the smoothing parameter in this section as the asymptotics depend on whether $\lambda$ is fixed or goes to zero asymptotically. In particular, we shall denote by $\lambda$ a fixed smoothing parameter and by $\lambda_{n}$ one dependent on the sample size.

We need the following assumptions to obtain consistency.
Assumption 1. $\left\{X_{i}, Y_{i}\right\}$ are iid random samples such that

$$
\begin{equation*}
Y_{i}=f\left(X_{i} ; \theta\right)+e_{i}=\beta_{0}+\beta_{1} \int_{0}^{X_{i}} \exp \left(-\int_{0}^{s} g\left(c^{T} \Phi(t)\right) d t\right) d s+e_{i} \tag{12}
\end{equation*}
$$

where $e_{i}$ 's are iid random errors with mean zero and finite variance $\sigma^{2}>0$.
Assumption 2. For all $x$, the conditional mean function $f(x ; \theta)$ is continuous in $\theta \in \Theta$, which is compact.

Assumption 3. (a) $\frac{1}{n} \sum_{i=1}^{n}\left\{f\left(x_{i} ; \theta^{*}\right)-f\left(x_{i} ; \theta\right)\right\}^{2}$ converges to some limit function uniformly in $\theta^{*}, \theta \in \Theta ;(b)$

$$
\mathcal{Q}(\theta)=\lim _{\theta} \frac{1}{n} \sum_{i=1}^{n}\left(f\left(X_{i} ; \theta\right)-f\left(X_{i} ; \theta^{0}\right)\right)^{2}
$$

has a unique minimum at $\theta=\theta^{0} \in \Theta$.
Theorem 1. Under assumptions 1-3, if the smooth parameter $\lambda_{n}=o(1)$, then a sequence of penalized least estimators minimizing the objective function (5) exists and $\hat{\theta}\left(\lambda_{n}\right) \xrightarrow{p} \theta^{0}$ as $n \rightarrow \infty$.

Remark 5. The variance of $\hat{\theta}\left(\lambda_{n}\right)$ goes to 0 as $n$ tends to $\infty$ whether or not $\lambda_{n}$ tends to 0. However, if $\lambda_{n} \rightarrow 0$ as $n \rightarrow \infty$, then the bias also tends to 0 and consistency can be established.

Next we derive the asymptotic normality. We first derive the asymptotics with $\lambda$ fixed. This is needed for finite sample inference. Let $W(\lambda)$ be a $n \times 2$ matrix with the $i$ th row $W_{i}=\left(1, m\left(X_{i} ; c(\lambda)\right)\right), i=1, \ldots, n$. Define

$$
\begin{align*}
P_{W}(\lambda) & =W(\lambda)\left(W(\lambda)^{T} W(\lambda)\right)^{-1} W(\lambda)^{T} \\
P_{Z}(\lambda) & =\left(\beta_{1}(\lambda) Z(\lambda)\right)\left(\beta_{1}^{2}(\lambda) Z(\lambda)^{T} Z(\lambda)+n \lambda D^{\prime \prime}\right)^{-1}\left(\beta_{1}(\lambda) Z^{T}(\lambda)\right) \tag{13}
\end{align*}
$$

and $\hat{W}(\lambda), \hat{P}_{W}(\lambda)$ and $\hat{P}_{Z}(\lambda)$ their sample analogs evaluated at $\hat{\theta}(\lambda)$, the penalized least squares estimators.

Under the assumption of iid errors, the variance $\sigma^{2}$ is estimated by the sum of squared residuals divided by proper degrees of freedom. Our semiparametric estimator has two parametric parameters $\beta_{0}$ and $\beta_{1}$, and a nonparametric smoother $m(X ; c)$. The degrees of freedom of the smoother, which can be viewed as its equivalent number of coefficients to that of a power series approximation, are calculated as $\operatorname{tr}\left(\hat{P}_{Z}(\lambda)\right)$. Therefore we estimate $\sigma^{2}$ with

$$
s^{2}=\frac{\sum_{i=1}^{n} \hat{e}_{i}^{2}}{n-\operatorname{tr}\left(\hat{P}_{Z}(\lambda)\right)-2} .
$$

Remark 6. Alternatively, we can use the degrees of freedom of the residuals in the calculation of variance. For linear smoothers, the residual degrees of freedom are given by $2 \operatorname{tr}\left(\hat{P}_{Z}(\lambda)\right)-\operatorname{tr}\left(\hat{P}_{Z}^{2}(\lambda)\right)$, cf. Ruppert et al. (2003) and references therein. In practice, these two specifications often give similar results.

We make the following assumptions for asymptotic normality.
Assumption 4. The following penalized objective function

$$
\mathcal{Q}_{\lambda}(\theta)=\mathcal{Q}(\theta)+\lambda D(f(\theta))
$$

has a unique minimum at $\theta(\lambda)$ in the interior of $\Theta$, where $\lambda$ is positive and finite.
Assumption 5. The conditional mean function $f(\cdot ; \theta)$ is twice continuously differentiable in a neighborhood of $\theta(\lambda)$, and $P_{W}(\lambda)$ and $P_{Z}(\lambda)$ converge uniformly in $\theta$ in a neighborhood of $\theta(\lambda)$.

Below we present an asymptotic normality result of the estimator. We choose to report results for the predicted values because the spline coefficients of the models usually are not of direct interest. We can construct confidence intervals for quantities of interest, for instance the marginal value of productivity in the estimation of production functions, based on the asymptotic properties of the estimators.

Theorem 2. Suppose that $\lambda$ is a fixed smoothing parameter. Under assumptions 1, 2, $3(a), 4$ and 5, a sequence of penalized spline estimators $\hat{\theta}(\lambda) \xrightarrow{p} \theta(\lambda)$ as $n \rightarrow \infty$. Denote
$Y(\lambda)=f(X ; \theta(\lambda))$ and $\hat{Y}(\lambda)=f(X ; \hat{\theta}(\lambda))$. Then $\sqrt{n}(\hat{Y}(\lambda)-Y(\lambda)) \xrightarrow{d} \mathcal{N}(0, V(\lambda))$ as $n \rightarrow \infty$, where

$$
\begin{equation*}
V(\lambda)=\sigma^{2}\left(P_{W}(\lambda)+P_{Z}^{2}(\lambda)\right) \tag{14}
\end{equation*}
$$

Define $\hat{V}(\lambda)=s^{2}\left(\hat{P}_{W}(\lambda)+\hat{P}_{Z}^{2}(\lambda)\right) . \hat{V}(\lambda) \xrightarrow{p} V(\lambda)$ as $n \rightarrow \infty$.
Denote by $\hat{V}_{i}(\lambda)$ the $i$ th diagonal element of $\hat{V}(\lambda)$. We construct the asymptotic $(1-\alpha) \%$ confidence interval of $\hat{Y}_{i}$ by

$$
\begin{equation*}
\hat{Y}_{i} \pm z_{1-\alpha / 2} \sqrt{\hat{V}_{i}(\lambda)} \tag{15}
\end{equation*}
$$

where $z_{1-\alpha / 2}$ is the critical value from the standard normal distribution at the confidence level $\alpha$.

Remark 7. The confidence interval (15) is about $Y(\lambda)=E[f(\cdot ; \hat{\theta}(\lambda)]$, the best projection, rather than $f\left(\cdot ; \theta^{0}\right)$. This is a well-known issue with series-based nonparametric estimations, of which the bias terms are generally not available. Although bias is inherent in nonparametric regression, approximate unbiasedness is often assumed and (15) can be interpreted as approximate confidence interval. Since this approximate confidence interval is oftentimes over optimistic, Hastie and Tibshirani (1990) suggested replacing $z_{1-\alpha / 2}$ in (15) with $t_{1-\alpha / 2, d f}$, where df is the proper degrees of freedom for nonparametric regressions. Eubank (1999) suggested Bonferroni methods to calculate confidence bands. Ruppert et al. (2003) discussed bias-corrected confidence intervals.

Remark 8. Our estimator is semiparametric with two parametric coefficients. Taking $\hat{m}$ as nuisance parameters, the estimator can be viewed as a two-step estimator with nonparametric first step estimates. Newey (1994) and Ai and Chen (2007) discussed the estimation of asymptotic semiparametric variance of the second stage estimates. Recently Ackerberg et al. (2012) showed that the asymptotic parametric variance that ignores the nonparametric nature of the first stage (for instance, the method of Newey (1984)) is numerically identical to the semiparametric variance. In particular, Ackerberg et al. (2012) provided several examples that use sieve estimators in the first step. The penalized spline estimator investigated in this study fits into their framework naturally.

Lastly, we derive the asymptotics with $\lambda_{n} \rightarrow 0$, corresponding to the limiting case where
the shrinkage bias is asymptotically negligible. Define $P_{W}^{0}=P_{W}\left(\theta^{0}\right)$ and $P_{Z}^{0}=P_{Z}\left(\theta^{0}\right)$ evaluated at $\lambda=0$. We can then establish the following result.

Theorem 3. Suppose that conditions 1, 2, 3 hold and conditions 4 and 5 hold with $\lambda=0$. If the smoothing parameter $\lambda_{n}=o\left(n^{-1 / 2}\right)$, then a sequence of penalized spline estimator $\hat{\theta}\left(\lambda_{n}\right) \xrightarrow{p} \theta^{0}$ as $n \rightarrow \infty$. Denote $\hat{Y}\left(\lambda_{n}\right)=f\left(X ; \hat{\theta}\left(\lambda_{n}\right)\right)$. Then $\sqrt{n}\left(\hat{Y}\left(\lambda_{n}\right)-Y\right) \xrightarrow{d} \mathcal{N}\left(0, V^{0}\right)$ as $n \rightarrow \infty$, where

$$
\begin{equation*}
V^{0}=\sigma^{2}\left(P_{W}^{0}+P_{Z}^{0}\right) . \tag{16}
\end{equation*}
$$

Remark 9. The limiting $P_{Z}\left(\lambda_{n}\right)$, defined in (13), is obtained by setting $\lambda_{n}=0$, yielding

$$
P_{Z}^{0}=Z\left(Z^{T} Z\right)^{-1} Z^{T}
$$

Since $P_{z}^{0}$ is now idempotent, we have $P_{Z}^{0}$ instead of $\left(P_{Z}^{0}\right)^{2}$ as in (14). For finite sample inference, one would expect $V^{0}$ overestimate the variance of $\hat{\theta}\left(\lambda_{n}\right)$ for a given $\lambda_{n}>0$.

## 5 Specification of Spline Basis and Smoothing Parameter

Implementation of the penalized spline estimators entails the specification of spline basis functions and smoothing parameters. The former includes the type of splines, number and location of knots. Commonly used splines include the truncated power series, $B$-splines and radial basis splines. The spline literature indicates that the practical differences among these splines are oftentimes quite small.

Because penalized spline estimations normally use a relatively generous spline basis, the number and location of knots play a relatively minor role in the estimation. We follow the automatic knot selection rule of Ruppert (2002), where the number of knots is given by

$$
\begin{equation*}
M=\min \left(\frac{1}{4} \times \text { number of unique } X, 35\right) \tag{17}
\end{equation*}
$$

and the knots are placed at the $m /(M+1)$-th sample quantile of the unique $X$ 's for $m=$ $1, \ldots, M$.

It is well known that spline estimators depend crucially on the smoothing parameter (cf. Ruppert, 2002). A commonly used approach for smoothing parameter selection is the method of cross validation (CV). Let $\hat{Y}_{(i)}$ be the prediction of $Y_{i}$ by a given estimator that uses all but the $i$ th observation. The 'leave-one-out' least squares cross validation criterion, in terms of sum of squared residuals, is given by

$$
C V=\sum_{i=1}^{n}\left(Y_{i}-\hat{Y}_{(i)}\right)^{2}
$$

Direct implementation of the cross validation is straightforward but often costly, especially for nonlinear nonparametric estimators without analytical solutions. For linear estimators, there exists an exact formula to evaluate the least squares cross validation criterion function, using only regression results based on the full sample. This exact solution usually does not exist for nonlinear estimations. Nonetheless, there exist approximate formulations that have been shown to give rather close results. Below we derive an approximate formula of the cross validation criterion for the proposed estimator. For $i=1, \ldots, n$, denote by $\hat{c}_{(i)}$ the solution to

$$
\frac{1}{n} \sum_{k=1, k \neq i}^{n}\left(Y_{k}-\beta_{0}-\beta_{1} m\left(X_{k} ; c\right)\right)^{2}+\lambda D(m(x))
$$

and $\hat{Y}_{(i)}$ be the prediction of $Y_{i}$ evaluated at $\hat{c}_{(i)}$. We establish the following result.
Theorem 4. Let $s_{i}$ be the ith diagonal element of $P_{Z}$ given in (13) and $\hat{s}_{i}$ its corresponding sample analog, $i=1, \ldots, n$. The Cross Validation (CV) criterion satisfies

$$
\begin{equation*}
C V=\sum_{i=1}^{n}\left(Y_{i}-\hat{Y}_{(i)}\right)^{2}=\sum_{i=1}^{n}\left(\frac{Y_{i}-\hat{Y}_{i}}{1-\hat{s}_{i}}\right)^{2}+o_{p}(1) \tag{18}
\end{equation*}
$$

Generalized Cross Validation (GCV) is a widely used and often more robust alternative to the CV criterion. It can be obtained by replacing $1-\hat{s}_{i}$ in (18) with $1-\frac{1}{n} \operatorname{tr}\left(\hat{P}_{Z}\right)$ (cf. Wahba, 1990). One can infer readily from Theorem 4 that in our case

$$
\mathrm{GCV} \approx \sum_{i=1}^{n}\left(\frac{Y_{i}-\hat{Y}_{i}}{1-\frac{1}{n} \operatorname{tr}\left(\hat{P}_{Z}\right)}\right)^{2}=\sum_{i=1}^{n}\left(\frac{Y_{i}-\hat{Y}_{i}}{1-\frac{1}{n} \sum_{i=1}^{n} \hat{s}_{i}}\right)^{2}
$$

Remark 10. An alternative criterion of smoothing parameter selection is the estimated risk (cf. Eubank 1999). Although conceptually simple, this criterion requires a proper prior estimate of $\sigma^{2}$. However, the optimal smoothing parameter for a conditional mean estimator generally is not optimal for the variance estimator. Another option is a likelihood based method that treats the spline coefficients as random coefficients. We model the spline coefficients as zero mean Gaussian processes and estimate using a mixed effect random coefficient model. Cf. Wand (2006) for an overview of this approach.

## 6 Multiple regressions

In this section we consider the case where $y$ is a function of $J(\geq 2)$ variables, being monotone and concave in each regressor. For multiple regressions, we adopt the convention that all notations, whenever necessary, are indexed by a subscript to make explicit their dependence on the specific coordinate $j=1, \ldots, J$. We focus on the case of the additive model:

$$
Y_{i}=\beta_{0}+\sum_{j=1}^{J} \beta_{j} m_{j}\left(X_{j, i}\right)+e_{i}, m_{j}^{\prime}>0 \text { and } m_{j}^{\prime \prime}<0
$$

For a general treatment of additive models, see Hastie and Tibshirani (1990).
We estimate the additive model using the penalized spline estimator by minimizing the following objective function:

$$
\frac{1}{n} \sum_{i=1}^{n}\left(Y_{i}-\beta_{0}-\sum_{j=1}^{J} \beta_{j} m_{j}\left(X_{j, i}\right)\right)^{2}+\sum_{j=1}^{J} \lambda_{j} D_{j}
$$

where $D_{j}=D\left(m_{j}(x)\right)$ and $\lambda_{j}$ is the corresponding smoothing parameter for $j=1, \ldots, J$. To ease the notational burden, we suppress the dependence of various quantities on $\lambda$ in this section. The Gauss-Jordan algorithm described above for the single covariate case can be extended to the multiple covariates case by updating the coefficients $c_{j}, j=1, \ldots, J$, sequentially via back-fitting. Alternatively, we can update all coefficients simultaneously for
possible efficiency gains. For $j, k \in(1, \ldots, J)$, define

$$
\hat{S}_{j}=\frac{1}{n} \hat{\beta}_{j} \hat{Z}_{j}^{T} \hat{e}-\lambda_{j} \hat{D}_{j}^{\prime},
$$

and

$$
\hat{R}_{j, k}= \begin{cases}\frac{1}{n} \hat{\beta}_{j}^{2} \hat{Z}_{j}^{T} \hat{Z}_{j}+\lambda_{j} \hat{D}_{j}^{\prime \prime}, & \text { if } j=k ; \\ \frac{1}{n} \hat{\beta}_{j} \hat{\beta}_{k} \hat{Z}_{j}^{T} \hat{Z}_{k}, & \text { if } j \neq k,\end{cases}
$$

where $\hat{Z}_{j}=\left(\hat{Z}_{j, 1}^{T}, \ldots, \hat{Z}_{j, n}^{T}\right)^{T}$ with $\hat{Z}_{j, i}=\partial m_{j}\left(X_{j, i} ; \hat{c}_{j}\right) / \partial c_{j}$. Further define $\hat{c}=\left(\hat{c}_{1}^{T}, \ldots, \hat{c}_{J}^{T}\right)^{T}$, $\hat{S}=\left(\hat{S}_{1}^{T}, \ldots, \hat{S}_{J}^{T}\right)^{T}$, and

$$
\hat{R}=\left[\begin{array}{ccc}
\hat{R}_{1,1} & \cdots & \hat{R}_{1, J} \\
\vdots & \ddots & \vdots \\
\hat{R}_{J, 1} & \cdots & \hat{R}_{J, J}
\end{array}\right]
$$

The coefficients $\hat{c}$ are then updated simultaneously according to

$$
\begin{equation*}
\hat{c}=\hat{c}_{-}-\hat{R}^{-1} \hat{S} \tag{19}
\end{equation*}
$$

Given the current estimate $\hat{c}, \beta=\left(\beta_{0}, \ldots, \beta_{J}\right)^{T}$ is calculated using the ordinary least squares estimator. This process is iterated to update $c$ and $\beta$ alternatively until convergence.

Next let $W$ be a $n$ by $J+1$ matrix with the $i$ th row $W_{i}=\left(1, m_{1}\left(X_{1 i}\right), \ldots, m_{J}\left(X_{J i}\right)\right)$ and $B=\left(\beta_{1} Z_{1}^{T}, \ldots, \beta_{J} Z_{J}^{T}\right)^{T}$. Define

$$
\begin{aligned}
P_{W} & =W\left(W^{T} W\right)^{-1} W^{T} \\
P_{Z} & =B^{T} R B
\end{aligned}
$$

where $R$ is defined analogously to $\hat{R}$. The residual variance is estimated by

$$
s^{2}=\frac{\sum_{i=1}^{n} \hat{e}^{2}}{n-1-J-\operatorname{tr}\left(\hat{P}_{Z}\right)} .
$$

The variance of the predictions of the additive model can then be calculated as

$$
\hat{V}=s^{2}\left(\hat{P}_{W}+\hat{P}_{Z}^{2}\right)
$$

A detailed investigation of the theoretical properties of the multiple regressions is beyond the scope of the current paper. We conjecture that under similar regularity conditions, the large sample properties derived in the previous section apply here (cf. Aerts, et al. (2002) for asymptotics of penalized spline estimators for additive models.)

## 7 Monte Carlo Simulations

In this section we use Monte Carlo simulations to assess the finite sample performance of our proposed estimator. We consider the following experiments:

- Experiment I:

$$
Y_{i}=f_{1}\left(X_{i}\right)+e_{i}=1+\log \left(0.1+X_{i}\right)+e_{i}
$$

- Experiment II:

$$
Y_{i}=f_{2}\left(X_{i}\right)+e_{i}=5-5 \times \exp \left(1-X_{i}\right)+e_{i}
$$

- Experiment III:

$$
\begin{aligned}
Y_{i} & =f_{21}\left(X_{1 i}\right)+f_{22}\left(X_{2 i}\right)+e_{i} \\
& =1+2 \times \log \left(0.01+X_{1 i}\right)+3 \times \log \left(0.01+X_{2 i}\right)+e_{i}
\end{aligned}
$$

In all three experiments, we set the sample size $n=100, X$ be iid random variables from the standard uniform distribution, and $e$ be iid random errors from the standard normal distribution. Each experiment is repeated 300 times. Experiments I and II study univariate monotone and concave functions, while Experiment III examines an additive function with two components, each being monotone and concave.

In each experiment, we estimate the underlying relationship using the proposed estimator. We use the cubic $B$-spline basis and the number and locations of knots are determined according to the automatic knot selection rule (17). We experiment with the CV, GCV and the likelihood based method of smoothing parameter selection. The results are quantitatively similar. To save space, we only report results based on the GCV.

For comparison, we consider two alternative estimators: the cubic smoothing spline estimator and the cubic polynomial estimator. The smoothing spline estimator is most flexible and does not impose any shape constraints. The cubic polynomial estimator represents the other extremum, which is the limiting case of cubic smoothing spline estimator when its smoothing parameter approaches infinity.

We use two criteria to gauge the performance of these competing estimators. For goodness-of-fit, we report the mean and median of the mean squared errors across all repetitions. To check their compliances with shape restrictions, we evaluate the first and second derivatives of the fitted curves for each observation and report the percentage of observationspecific monotonicity and concavity of the fitted curves evaluated at sample values.

Table 1: Simulation Results

|  | Estimator | Experiment I | Experiment II | Experiment III |
| :---: | :---: | :---: | :---: | :---: |
| Mean-MSE | S-Spline | 307 | 336 | 1811 |
|  | Polynomial | 360 | 406 | 2616 |
|  | Spline | 388 | 366 | 1907 |
| Median-MSE | S-Spline | 254 | 250 | 1629 |
|  | Polynomial | 314 | 324 | 2549 |
|  | Spline | 331 | 277 | 1846 |
| Monotonicity (\%) | S-Spline | 100 | 100 | 100 |
|  | Polynomial | 93 | 96 | 99 |
|  | Spline | 95 | 98 | 92 |
| Concavity (\%) | S-Spline | 100 | 100 | 100 |
|  | Polynomial | 70 | 66 | 69 |
|  | Spline | 51 | 51 | 66 |

Denote by 'S-Spline' the shape-restricted semiparametric estimator, and by 'Polynomial' and 'Spline' the cubic polynomial and smoothing spline estimators respectively. Table 1 reports the simulation results. The S-Spline outperforms the other two estimators in all three experiments in terms of mean-MSE and median-MSE. By construction, monotonicity and concavity are satisfied globally under the S-Spline. For the other two estimators, we calculate their first and second order derivatives numerically on each data point. In Experiment III, the monotonicity and concavity percentages are reported separately for the two additive components. Our results show that monotonicity is satisfied in most cases, while the percentages of estimates satisfying concavity range from 50 to 70 percent. This is not
unexpected considering that higher order derivatives are generally more difficult to estimate.
Some illustrative results of Experiment I are presented in Figure 1. The left panel reports a typical picture of the regression results. The black curve is the constrained estimate, which is monotone and concave. The red line depicts the polynomial estimate, which appears to be concave on the observed range but is not monotone increasing near the right end. The smoothing splines estimate, represented by a blue line, is the most flexible and exhibits multiple violations of monotonicity and concavity. The right panel plots one estimated curve by the constrained estimator, along with its approximate $95 \%$ variation bound indicated by the red lines. Also reported is the bootstrapped variation bound, based on 100 re-sampled estimates, in blue lines. One can see that the asymptotic bound closely tracks that produced by a bootstrap procedure, which is computationally more expensive.


Figure 1: Left: estimated curves (black: S-Spline; red: Polynomial; blue: Spline); Right: $95 \%$ variation bound about a fitted curve (red: asymptotic; blue: bootstrap)

## 8 Empirical applications

In this section, we present two illustrative applications of the proposed method. The first application investigates the relationship between revealed risk attitude and optimism. The data come from a survey conducted by Mansour et al. (2008). In this survey, participants were offered the opportunity to enter a heads-and-tails game. A coin is flipped ten times; each time a head appears, the participant receives 10 euros. The participant is then asked to estimate the number of times heads will occur. The participant is also asked to reveal the maximum amount she is willing to pay (WTP) in order to take part in this game. The aim of this experiment is to obtain measures of individual levels of optimism and risk aversion. The sample has $n=1,536$ observations. Summary statistics of the data are reported in the top panel of Table 2. On average, the participants are pessimistic (the average expectation 3.9 is less than 5 , the unbiased expectation) and risk averse (the average WTP 16.3 is below the fair expectation 50 and also below 39, which is the expected risk neutral WTP given the average expectation of 3.9).

Table 2: Summary statistics

|  | Mean | S.D. | Min. | Max. |
| :--- | ---: | ---: | ---: | ---: |
| Risk and Optimism |  |  |  |  |
| Optimism | 3.9 | 1.8 | 0 | 10 |
| WTP | 12.0 | 13.6 | 0 | 100 |
| Production Data |  |  |  |  |
| Output | 16.3 | 8.3 | 1.7 | 37.1 |
| Capital | 4.8 | 2.8 | 9.6 | 0.3 |
| Labor | 57.7 | 27.2 | 1.1 | 98.9 |

For $i=1, \ldots, n$, let $Y_{i}$ be individual $i$ 's estimation of the number of heads, and $X_{i}$ her maximum willingness to pay. We are interested in estimating the relationship between these two measures. According to preference and utility function theories, there exists a monotone relationship between risk aversion and optimism (see Mansour et al. (2008) and references therein). Taking the WTP as a proxy for degree of risk aversion or risk loving, one expects a monotone increasing relationship between $Y_{i}$ and $X_{i}$. Since measures of optimism are naturally bounded from above by 10 , we expect the $Y_{i}$ as a function of $X_{i}$ to level off
as $X_{i}$ gets large (there is no upper bound for $X_{i}$; but as expected, no participants offered more than 100 euros). Therefore, it is plausible that $Y=f(X)$ is monotone increasing and concave.


Figure 2: Risk tolerance vs optimism: data and estimates (shaded areas represent $95 \%$ variation bounds)

The upper right plot of Figure 2 shows the participants' answers to the two questions, clearly implying a monotone and possibly concave relationship between these two measures.

Thus in our illustration, we apply the proposed method to the following model:

$$
Y_{i}=f\left(X_{i}\right)+e_{i}, i=1, \ldots, n
$$

where $f^{\prime}>0, f^{\prime \prime}<0$, and $e_{i}$ are iid errors with mean zero and finite variance. For comparison, we also estimate the model using the cubic polynomial estimator and the cubic smoothing spline estimator. The estimation results are reported Figure 2. All three estimators capture the general patterns of the data. Also plotted are the $95 \%$ confidence intervals.

The smoothing spline estimate clearly fails to be monotone increasing. A close examination of the data indicates that there are several possible outliers with low degrees of optimism but high WTP, suggesting inconsistency in the preferences of these participants. These outliers appear to exert a disproportionately large influence on the smoothing spline estimate. The polynomial estimate is closer to the constrained estimates, but is not concave. The slight acceleration in optimism near the top end of the WTP range does not seem to be supported by the data. This spurious pattern most likely are due to oscillations typically associated with 'global' projection estimators, particularly power series estimators, where performance of fitting in one region of the curves may affect that capability in a different region. In contrast, local smoothers, such as kernel or spline estimators, do not suffer these types of global oscillations.

Across all three estimates, the confidence intervals are tighter for small values of WTP and gradually increase with WTP, largely due to number of observations falling rapidly as WTP rises. Nonetheless, the constrained estimate exhibit the smallest variance near the upper boundary, suggesting its robustness against potential outliers and sparsity of observations.

The second example concerns the estimation of a production function. According to economic theory, production functions are monotone increasing and concave with respect to inputs (cf. Diewert and Wales (1987)). We use the benchmark data in Coelli (1996), which contains information on the level of output and capital and labor inputs of 60 firms. The bottom panel of Table 2 reports summary statistics of the data set.

We assume that the production function takes the following additive form:

$$
Q_{i}=f_{1}\left(C_{i}\right)+f_{2}\left(L_{i}\right)+e_{i}, i=1, \ldots, n
$$

where $Q, C$ and $L$ denote output, capital and labor respectively, $e_{i}$ are iid errors with mean zero and finite variance, and $n=60$. We also assume that $f_{j}^{\prime}>0$ and $f_{j}^{\prime \prime}<0$ for $j=$ 1,2 . As in the previous example, we estimate the model using the constrained estimator, the polynomial estimator and the smoothing spline estimator. The estimation results are reported in Figure 3. The right panel plots the estimated surface and the left panel their corresponding contours. The general shape captured by the three estimators is similar. However, the monotonicity condition is clearly violated in the polynomial and smoothing spline estimates.

## 9 Concluding Remarks

We have proposed a semiparametric estimator that accommodates shape restrictions such as monotonicity and concavity. Our method employs an integral transformation to achieve the desired shape constraints. The resulting estimates satisfy the constraints globally. We use penalized splines to achieve flexibility while maintaining shape constraints. We have proposed an iterative algorithm and a cross validation criterion for smoothing parameter selection. We have derived the asymptotic variance of the proposed estimator and have further extended the proposed method to multiple regressions under the framework of additive models. Our Monte Carlo simulations and two empirical examples illustrate the appeal of the estimator in terms of its finite sample performance and its usefulness in capturing the shape restrictions while also providing relative flexibility in fitting the nonlinear relationships we have estimated.

We conclude by suggesting some possible generalizations of the proposed method. First, the current model considers continuous outcomes. Generalization to discrete or range-limited variables in the framework of the generalized linear models, as in Shively et al. (2011), is a natural extension of the approach we have taken. Second, we envision that our methods can be generalized to accommodate inter-temporally or spatially correlated errors, or com-


Figure 3: Estimated production function (Top: Constrained estimate; Middle: Polynomial estimate; Bottom: Smoothing spline estimate)
posite errors as in the case of panel data analysis. Third, we restrict ourselves to additive models in this study. Relaxations of this restriction to accommodate interactions or more general non-separable structures while maintaining shape constraints may be of interest for future research. Lastly, we acknowledge that it is desirable to be able to test the validity of constraints implied by economic theories. Heckmam and Ramsay (2000) presented the $L$ spline estimators, whose model-based penalties are defined via linear differential functions. Their method provides a natural framework to test the validity of constraints implied by differential equations, such as those used in our estimator.

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## Appendix

Proof of Theorem 1. We can rewrite the objective function as

$$
\begin{aligned}
Q_{\lambda_{n}}(\theta) & =\frac{1}{n} \sum_{i=1}^{n}\left\{Y_{i}-f\left(X_{i} ; \theta^{0}\right)+f\left(X_{i} ; \theta\right)-f\left(X_{i} ; \theta^{0}\right)\right\}^{2}+\lambda_{n} D(f(\theta)) \\
& =\frac{1}{n} \sum_{i=1}^{n} e_{i}^{2}+\frac{2}{n} \sum_{i=1}^{n}\left\{f\left(X_{i} ; \theta^{0}\right)-f\left(X_{i} ; \theta\right)\right\}^{2} e_{i}+\frac{1}{n} \sum_{i=1}^{n}\left\{f\left(X_{i} ; \theta^{0}\right)-f\left(X_{i} ; \theta\right)\right\}^{2}+\lambda_{n} D(f(\theta)) .
\end{aligned}
$$

Under assumptions 1,2 , and 3 a, the first and third terms converge to $\sigma^{2}$ and $\mathcal{Q}(\theta)$ respectively, and the second term converges to zero. In addition, the last term vanishes if $\lambda_{n} \rightarrow 0$. It follows that

$$
Q_{\lambda}\left(\theta_{n}\right) \xrightarrow{p} \mathcal{Q}(\theta)+\sigma^{2}
$$

if $\lambda_{n}=o(1)$.

Next let $\hat{\theta}\left(\lambda_{n}\right)$ be the penalized least square estimators. It follows that

$$
Q_{\lambda_{n}}\left(\hat{\theta}\left(\lambda_{n}\right)\right) \leq Q_{\lambda_{n}}\left(\theta^{0}\right)
$$

Under assumption 3.a, the left hand side converges to, say, $\mathcal{Q}\left(\theta^{\prime}\right)+\sigma^{2}, \theta^{\prime} \in \Theta$. It follows that

$$
\mathcal{Q}\left(\theta^{\prime}\right)+\sigma^{2} \leq \mathcal{Q}\left(\theta^{0}\right)+\sigma^{2}=\sigma^{2}
$$

implying $Q\left(\theta^{\prime}\right)=0$. Thus under assumption 3a and 3 b , we have $\theta^{\prime}=\theta^{0}$, which establishes the consistency of the penalized least square estimator.

Proof of Theorem 2. Rewrite

$$
\hat{Y}(\lambda)-Y(\lambda)=\{\hat{W}(\lambda)-W(\lambda)\} \hat{\beta}(\lambda)+W(\lambda)(\hat{\beta}(\lambda)-\beta(\lambda))
$$

It follows that

$$
\begin{align*}
\operatorname{Var}(\hat{Y}(\lambda))= & \operatorname{Var}((\hat{W}(\lambda)-W(\lambda)) \hat{\beta}(\lambda))+\operatorname{Var}(W(\lambda)(\hat{\beta}(\lambda)-\beta(\lambda))) \\
& +2 \operatorname{cov}((\hat{W}(\lambda)-W(\lambda)) \hat{\beta}(\lambda), W(\lambda)(\hat{\beta}(\lambda)-\beta(\lambda))) \tag{A.1}
\end{align*}
$$

First note that the third term vanishes asymptotically. Since $\beta(\lambda)=\left(W(\lambda)^{T} W(\lambda)\right)^{-1} W(\lambda) Y(\lambda)$, it follows readily that

$$
\begin{equation*}
\operatorname{Var}(W(\lambda)(\hat{\beta}(\lambda)-\beta(\lambda)))=\sigma^{2} P_{W}(\lambda) \tag{A.2}
\end{equation*}
$$

From (11), we have under assumption 5 that

$$
\operatorname{Var}(\sqrt{n}(\hat{c}(\lambda)-c(\lambda))=\Omega(\lambda)
$$

with

$$
\Omega(\lambda)=\sigma^{2}\left(\beta_{1}(\lambda) Z(\lambda)\left(\beta_{1}^{2}(\lambda) Z(\lambda)^{T} Z(\lambda)+n \lambda D^{\prime \prime}\right)^{-2}\left(\beta_{1}(\lambda) Z^{T}(\lambda)\right)\right.
$$

Next note that

$$
\begin{aligned}
(\hat{W}(\lambda)-W(\lambda)) \hat{\beta}(\lambda) & =(\hat{W}(\lambda)-W(\lambda)) \beta(\lambda)+o_{p}(1) \\
& =\beta_{1}(\lambda)(m(X ; \hat{c}(\lambda))-m(X ; c(\lambda)))+o_{p}(1) \\
& =\beta_{1}(\lambda) Z(\lambda)(\hat{c}(\lambda)-c(\lambda))+o_{p}(1)
\end{aligned}
$$

It follows that

$$
\begin{equation*}
\operatorname{Var}((\hat{W}(\lambda)-W(\lambda)) \hat{\beta}(\lambda))=\left(\beta_{1} Z^{T}(\lambda)\right) \Omega(\lambda)\left(\beta_{1}(\lambda) Z^{T}(\lambda)\right)=\sigma^{2} P_{Z}^{2}(\lambda) \tag{A.3}
\end{equation*}
$$

Combining (A.2) and (A.3) then yields (14). Under assumptions 1, 2, 3(a), 4 and 5, the asymptotic normality can be readily established under the central limite theorem.

Lastly the variance of the error terms is estimated by $\left(\sum_{i=1}^{n} \hat{e}_{i}^{2}\right) /($ d.o.f. $)$, where the degrees of freedom is given by $n$ subtracted the effective number of parameters. The proposed semiparametric estimator has two parametric parameters, and the effective number of parameters (rank of the smoother) for the nonparametric part is calculated as $\operatorname{tr}\left(\hat{P}_{Z}\right)(\lambda)$ (Cf. Ruppert et al. (2003)). It follows that $s^{2} \xrightarrow{p} \sigma^{2}$ as $n \rightarrow \infty$. In addition, it is straightforward to show that $\hat{\beta}(\lambda), \hat{P}_{W}(\lambda)$ and $\hat{P}_{Z}(\lambda)$ converge in probability to $\beta(\lambda), P_{W}(\lambda)$ and $P_{Z}(\lambda)$ as $n \rightarrow \infty$. It follows that under assumption $5, \hat{V}(\lambda) \xrightarrow{p} V(\lambda)$ as $n \rightarrow \infty$, which completes the proof of this theorem.

Proof of Theorem 3. From (11) we have

$$
\begin{aligned}
\hat{c}\left(\lambda_{n}\right) & =\left(\frac{1}{n} \hat{\beta}_{1}^{2} \hat{Z}^{T} \hat{Z}+\lambda_{n} \hat{D}^{\prime \prime}\right)^{-1}\left(\frac{1}{n} \hat{\beta}_{1} \hat{Z}^{T} \tilde{Y}+\lambda_{n}\left(\hat{D}^{\prime}-\hat{D}^{\prime \prime} \hat{c}_{-}\right)\right) \\
& =\left(\frac{1}{n} \hat{\beta}_{1}^{2} \hat{Z}^{T} \hat{Z}+\lambda_{n} \hat{D}^{\prime \prime}\right)^{-1}\left(\frac{1}{n} \hat{\beta}_{1} \hat{Z}^{T} \tilde{Y}+o_{p}(1)\right) \\
& \equiv\left(\frac{1}{n} B^{T} B+\lambda_{n} \hat{D}^{\prime \prime}\right)^{-1}\left(\frac{1}{n} B^{T} \tilde{Y}+o_{p}(1)\right) .
\end{aligned}
$$

A Taylor expansion of the above with respect to $\lambda_{n}$ around zero, using that $(I+\lambda A)^{-1}=$
$I-\lambda A+o(\lambda A)$ as $\lambda \rightarrow 0$ yields

$$
\begin{aligned}
\hat{c}\left(\lambda_{n}\right) & =\left(\left(\frac{1}{n} B^{T} B\right)^{-1} \frac{1}{n} B^{T} B+\lambda_{n} \hat{D}^{\prime \prime}\right)^{-1}\left(\frac{1}{n} B^{T} B\right)^{-1}\left(\frac{1}{n} B^{T} \tilde{Y}+o_{p}(1)\right) \\
& =\left(I-\lambda_{n} \hat{D}^{\prime \prime}+o\left(\lambda_{n} \hat{D}^{\prime \prime}\right)\right)\left(\frac{1}{n} B^{T} B\right)^{-1}\left(\frac{1}{n} B^{T} \tilde{Y}+o_{p}(1)\right) \\
& =\left(B^{T} B\right)^{-1} B^{T} \tilde{Y}+o\left(\lambda_{n} \hat{D}^{\prime \prime}\right)+o_{p}(1) \\
& =c^{0}+o\left(\lambda_{n} \hat{D}^{\prime \prime}\right)+o_{p}(1),
\end{aligned}
$$

where the last equality is due to the consistency of $\hat{c}\left(\lambda_{n}\right)$ as $\lambda_{n} \rightarrow 0$ given in Theorem 1.
Next we can show that the variance of $\hat{c}\left(\lambda_{n}\right)$ is of order $\sigma^{2} / n$. It follows that $\operatorname{MSE}\left(\hat{c}\left(\lambda_{n}\right)\right)=$ $O_{p}\left(\sigma^{2} / n+\lambda_{n}^{2}\right)$ for bounded $\hat{D}^{\prime \prime}$ (which is implied by the compactness of $\Theta$ ). Thus for the asymptotic bias to vanish, we need $\lambda_{n}=o\left(n^{-1 / 2}\right)$. The asymptotic normality of the limiting case can then be established using essentially the same proof as for Theorem 2 and replacing the fixed $\lambda$ with zero, the limiting value of $\lambda_{n}$.

Proof of Theorem 4. Let $\hat{c}(i, w)$ be the solution to the following optimization

$$
\begin{equation*}
\left(w-\beta_{0}-\beta_{1} f\left(X_{i}\right)\right)^{2}+\sum_{k=1, k \neq i}^{n}\left(Y_{k}-\beta_{0}-\beta_{1} f\left(X_{k}\right)\right)^{2}+\lambda D(f(x)) \tag{A.4}
\end{equation*}
$$

It follows that $\hat{c}\left(i, \hat{Y}_{(i)}\right)=\hat{c}_{(i)}$.
Let $\Delta_{(i)}$ be an $n \times 1$ vector of zeros except that the $i$ th element equals $\hat{Y}_{(i)}-Y_{i}$. We can then write

$$
\hat{c}_{(i)}=\left(\hat{\beta}_{1}^{2} \hat{Z}^{T} \hat{Z}+\lambda \int_{\mathcal{X}} D^{\prime \prime}(x) d x\right)^{-1} \hat{\beta}_{1} \hat{Z}^{T}\left(\tilde{Y}+\Delta_{(i)}\right) .
$$

It follows that

$$
\begin{align*}
\tilde{Y}_{(i)}= & \hat{\beta}_{1} \hat{Z}_{i}^{T} \hat{c}_{(i)} \\
= & \hat{\beta}_{1} \hat{Z}_{i}^{T}\left(\hat{\beta}_{1}^{2} \hat{Z}^{T} \hat{Z}+\lambda \int_{\mathcal{X}} D^{\prime \prime}(x) d x\right)^{-1} \hat{\beta}_{1} \hat{Z}^{T} \tilde{Y} \\
& +\hat{\beta}_{1} \hat{Z}_{i}^{T}\left(\hat{\beta}_{1}^{2} \hat{Z}^{T} \hat{Z}+\lambda \int_{\mathcal{X}} D^{\prime \prime}(x) d x\right)^{-1} \hat{\beta}_{1} \hat{Z}^{T} \Delta_{(i)} \\
= & \tilde{Y}_{i}+s_{i}\left(\hat{Y}_{(i)}-Y_{i}\right) . \tag{A.5}
\end{align*}
$$

Next we use the Taylor approximation on $\hat{Y}_{(i)}$ to obtain

$$
\begin{aligned}
\tilde{Y}_{(i)} & =\hat{Y}_{(i)}-\hat{\beta}_{0}-\hat{\beta}_{1} f\left(X_{i} ; \hat{c}\right)-\hat{\beta}_{1} \hat{Z}_{i}^{T}\left(\hat{c}_{(i)}-\hat{c}\right)+\hat{\beta}_{1} \hat{Z}_{i}^{T} \hat{c}_{(i)}+o_{p}(1) \\
& =\hat{Y}_{(i)}-\hat{\beta}_{0}-\hat{\beta}_{1} f\left(X_{i} ; \hat{c}\right)+\hat{\beta}_{1} \hat{Z}_{i}^{T} \hat{c}+o_{p}(1) .
\end{aligned}
$$

It follows that

$$
\begin{equation*}
\tilde{Y}_{(i)}-\tilde{Y}_{i}=\hat{Y}_{(i)}-\hat{Y}_{i}+o_{p}(1) \tag{A.6}
\end{equation*}
$$

Plugging (A.6) into (A.5) and rearranging terms yields

$$
Y_{i}-\hat{Y}_{(i)}=\frac{Y_{i}-\hat{Y}_{i}}{1-s_{i}}+o_{p}(1),
$$

which gives (A.4) readily.


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[^1]:    ${ }^{1}$ The quantity $g$ reflects the relative curvature of $f$. Interestingly, we note that this is also the parameterization used to derive Arrow-Pratt utility.

[^2]:    ${ }^{2}$ Kneip, Sickles, and Soing (2012) used such penalized splines in their general treatment of nonparametric time varying and cross-sectionally heterogeneous panel estimator.

[^3]:    ${ }^{3}$ An alternative to the penalized spline method is the regression splines method, which balances the goodness-of-fit and smoothness trade-off through judicious selection of spline basis functions. The selection of basis functions for regression splines can be a daunting task, especially in multiple regressions. Consider a candidate set of $P$ basis functions. A complete subset selection, which exhausts all possible combinations of the basis functions, entails $2^{P}$ evaluations of candidate models.

[^4]:    ${ }^{4}$ Smoothing splines are a special case of penalized splines when the number of basis functions equals the number of unique observations. For a general treatment of smoothing splines, cf. Wahba (1990).

